Lesson 3: Exponents That Are Unit Fractions

- Let’s explore exponents like $\frac{1}{2}$ and $\frac{1}{3}$.

3.1: Sometimes It’s Squared and Sometimes It’s Cubed

Find a solution to each equation.

1. $x^2 = 25$

2. $z^2 = 7$

3. $y^3 = 8$

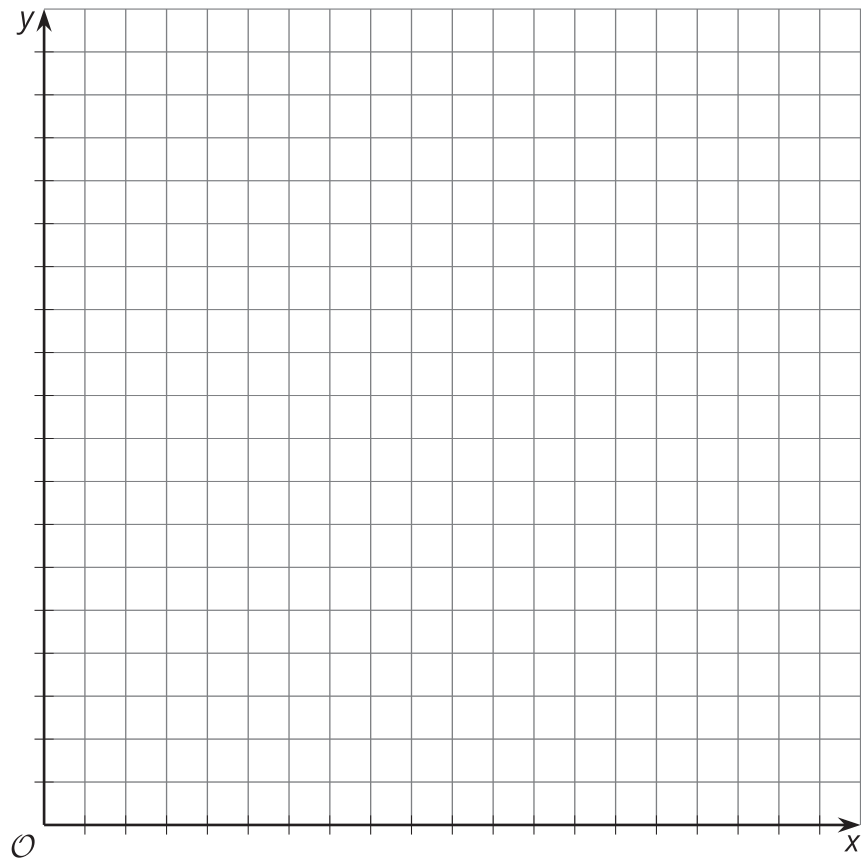
4. $w^3 = 19$

3.2: To the...Half?

1. Clare said, "I know that $9^2 = 9 \cdot 9$, $9^1 = 9$, and $9^0 = 1$. I wonder what $9^{\frac{1}{2}}$ means?"

   First, she graphed $y = 9^x$ for some whole number values of x, and estimated $9^{\frac{1}{2}}$ from the graph.

   a. Graph the function yourself. What estimate do you get for $9^{\frac{1}{2}}$?

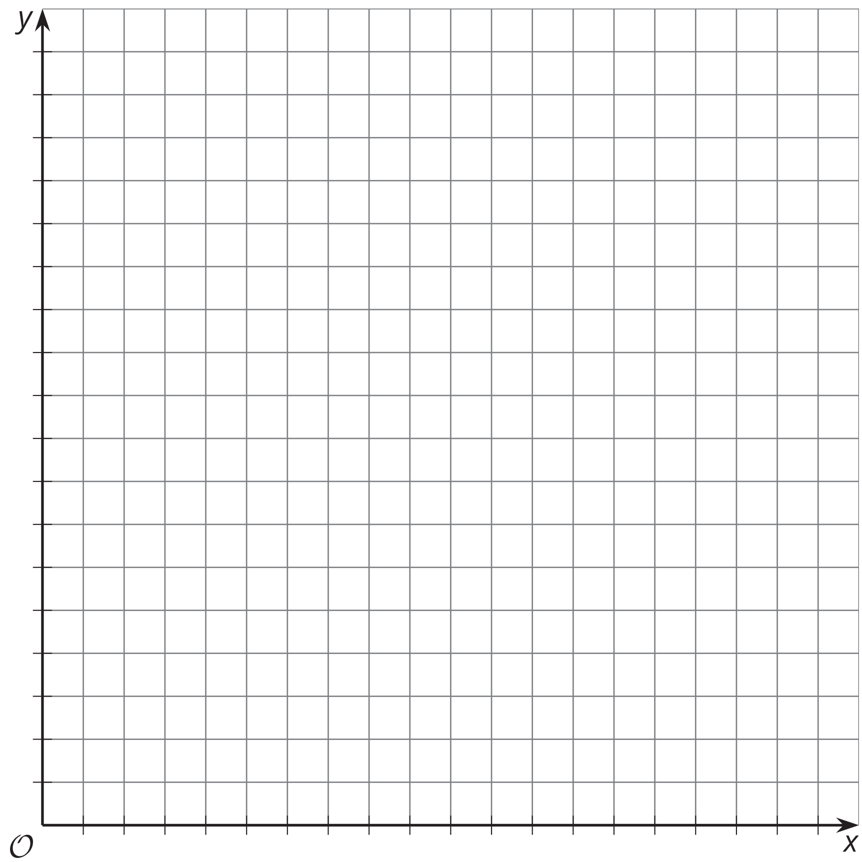


   b. Using the properties of exponents, Clare evaluated $9^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$. What did she get?

   c. For that to be true, what must the value of $9^{\frac{1}{2}}$ be?

2. Diego saw Clare’s work and said, “Now I’m wondering about $3^{\frac{1}{2}}$.” First he graphed $y = 3^x$ for some whole number values of x, and estimated $3^{\frac{1}{2}}$ from the graph.

   a. Graph the function yourself. What estimate do you get for $3^{\frac{1}{2}}$?



   b. Next he used exponent rules to find the value of $\left(3^{\frac{1}{2}}\right)^2$. What did he find?

   c. Then he said, “That looks like a root!” What do you think he means?

3.3: Fraction of What, Exactly?

Use the exponent rules and your understanding of roots to find the exact value of:

1. $25^{\frac{1}{2}}$

2. $15^{\frac{1}{2}}$

3. $8^{\frac{1}{3}}$

4. $2^{\frac{1}{3}}$

3.4: Exponents and Radicals

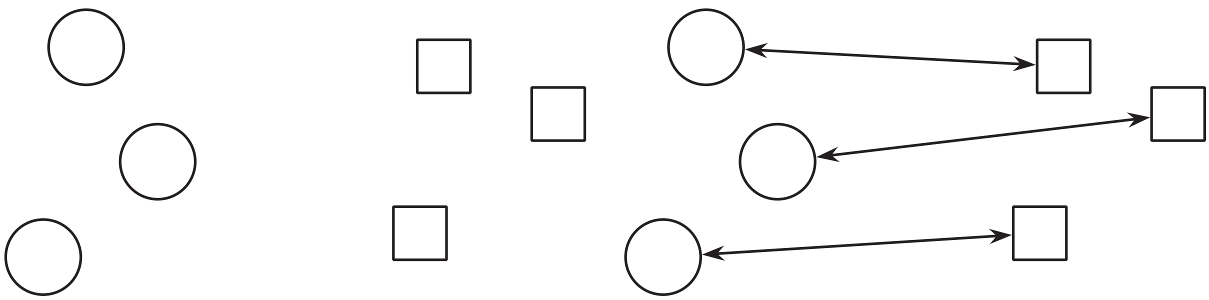
Match each exponential expression to an equivalent expression.

- 7^3
- 7^2
- 7^1
- 7^0
- 7^{-1}
- 7^{-2}
- 7^{-3}
- $7^{\frac{1}{2}}$
- $7^{-\frac{1}{2}}$
- $7^{\frac{1}{3}}$
- $7^{-\frac{1}{3}}$

- $\frac{1}{49}$
- $\frac{1}{343}$
- $\sqrt{7}$
- $\frac{1}{\sqrt[3]{7}}$
- $\sqrt[3]{7}$
- 49
- $\frac{1}{\sqrt{7}}$
- 343
- 7
- $\frac{1}{7}$
- 1

Are you ready for more?

How do we know without counting that the number of circles equals the number of squares? Because we can match every circle to exactly one square, and each square has a match:



We say that we have shown that there is a one-to-one correspondence of the set of circles and the set of squares. We can do this with infinite sets, too! For example, there are the same "number" of positive integers as there are even positive integers:

1	2	3	4	5	6	7	...
↕	↕	↕	↕	↕	↕	↕	
2	4	6	8	10	12	14	...

Every positive integer is matched to exactly one even positive integer, and every even integer has a match! We have shown that there is a one-to-one correspondence between the set of positive integers and the set of even positive integers. Whenever we can make a one-to-one matching like this of the positive integers to another set, we say the other set is *countable*.

1. Show that the set of square roots of positive integers is countable.

2. Show that the set of positive integer roots of 2 is countable.

3. Show that the set of positive integer roots of positive integers is countable. (Hint: there is a famous proof that the positive rational numbers are countable. Find and study this proof.)

Lesson 3 Summary

How can we make sense of the expression $11^{\frac{1}{2}}$? For this expression to make any sense at all, we should be able to apply exponent rules to it. Let's try squaring $11^{\frac{1}{2}}$ using exponent rules: $\left(11^{\frac{1}{2}}\right)^2 = 11^{\frac{1}{2} \cdot 2}$, which is simply 11. In other words, if we square the number $11^{\frac{1}{2}}$ using exponent rules, we get 11. That means that $11^{\frac{1}{2}}$ must be equal to $\sqrt{11}$.

Similarly, $11^{\frac{1}{3}}$ must be equal to $\sqrt[3]{11}$ because

$$\begin{aligned} \left(11^{\frac{1}{3}}\right)^3 &= 11^{\frac{1}{3} \cdot 3} \\ &= 11 \end{aligned}$$

In general, if a is any positive number, then

$$a^{\frac{1}{2}} = \sqrt{a}$$

and

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

Remember, these expressions that involve the $\sqrt{}$ symbol are often referred to as *radical* expressions.

Lesson 3 Practice Problems

1. Complete the table. Use powers of 64 in the top row and radicals or rational numbers in the bottom row.

64^1	$64^{\frac{1}{2}}$		64^0		64^{-1}
64		4		$\frac{1}{8}$	

2. Suppose that a friend missed class and never learned what $25^{\frac{1}{2}}$ means.

   a. Use exponent rules your friend would already know to calculate $25^{\frac{1}{2}} \cdot 25^{\frac{1}{2}}$.

   b. Explain why this means that $25^{\frac{1}{2}} = 5$.

3. Which expression is equivalent to $16^{\frac{1}{2}}$?

   A. $\frac{1}{4}$

   B. 4

   C. 8

   D. 16.5

4. Select **all** the expressions equivalent to 4^{10}.

A. $2^5 \cdot 2^2$

B. 2^{20}

C. $4^4 \cdot 4^6$

D. $4^7 \cdot 4^{-3}$

E. $\frac{4^4}{4^{-6}}$

(From Unit 3, Lesson 1.)

5. The table shows the edge length and volume of several different cubes. Complete the table using exact values.

edge length (ft)	3			$\sqrt[3]{100}$		$\sqrt[3]{147}$
volume (ft^3)		64	85		125	

(From Unit 3, Lesson 2.)

6. A square has side length $\sqrt{82}$ cm. What is the area of the square?

A. 9.05 cm^2

B. 82 cm^2

C. 164 cm^2

D. 6724 cm^2

(From Unit 3, Lesson 2.)

Lesson 4: Positive Rational Exponents

- Let's use roots to write exponents that are fractions.

4.1: Math Talk: Regrouping Fractions

Find the value of each expression mentally.

$\frac{1}{2} \cdot 5 \cdot 4$

$\frac{5}{2} \cdot 4$

$\frac{2}{3} \cdot 7 \cdot \frac{3}{2}$

$7 \cdot \frac{5}{3} \cdot \frac{3}{7}$

4.2: You Can Use Any Fraction As an Exponent

1. Use exponent rules to explain why these expressions are equal to each other:

   $\left(5^{\frac{1}{3}}\right)^2 \quad \left(5^2\right)^{\frac{1}{3}}$

2. Write $5^{\frac{2}{3}}$ using radicals.

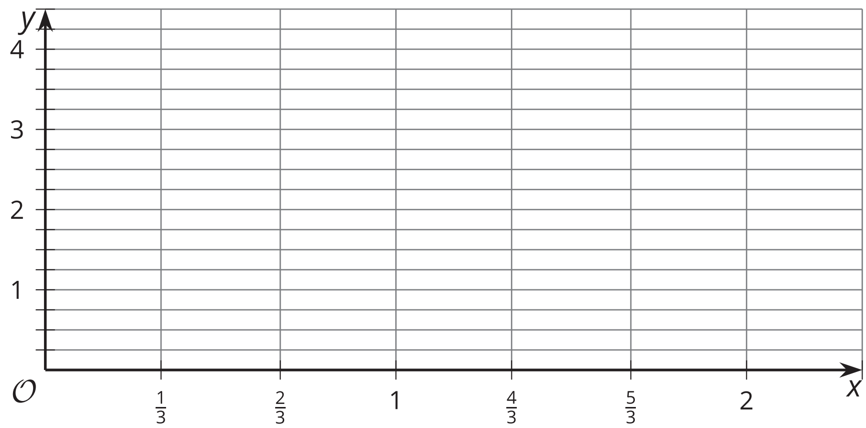
3. Write $5^{\frac{4}{3}}$ using radicals. Show your reasoning using exponent rules.

4.3: Fractional Powers Are Just Numbers

1. Complete the table as much as you can without using a calculator. (You should be able to fill in three spaces.)

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
2^x (using exponents)	2^0	$2^{\frac{1}{3}}$	$2^{\frac{2}{3}}$	2^1	$2^{\frac{4}{3}}$	$2^{\frac{5}{3}}$	2^2
2^x (decimal approximation)							

a. Plot the points that you filled in.



b. Connect the points as smoothly as you can.

c. Use this graph of $y = 2^x$ to estimate the value of the other powers in the table, and write your estimates in the table.

2. Let's investigate $2^{\frac{1}{3}}$:

a. Write $2^{\frac{1}{3}}$ using radical notation.

b. What is $\left(2^{\frac{1}{3}}\right)^3$?

c. Raise your estimate from the table of $2^{\frac{1}{3}}$ to the third power. What should it be? How close did you get?

3. Let's investigate $2^{\frac{2}{3}}$:

a. Write $2^{\frac{2}{3}}$ using radical notation.

b. What is the value of $\left(2^{\frac{2}{3}}\right)^3$?

c. Raise your estimate from the table of $2^{\frac{2}{3}}$ to the third power. What should it be? How close did you get?

Are you ready for more?

Answer these questions using the fact that $(1.26)^3 = 2.000376$.

1. Explain why $\sqrt[3]{2}$ is very close to 1.26. Is it larger or smaller than 1.26?

2. Is it possible to write $\sqrt[3]{2}$ exactly with a finite decimal expansion? Explain how you know.

Lesson 4 Summary

Using exponent rules, we know $3^{\frac{1}{4}}$ is the same as $\sqrt[4]{3}$ because $\left(3^{\frac{1}{4}}\right)^4 = 3$. But what about $3^{\frac{5}{4}}$?

Using exponent rules,

$$3^{\frac{5}{4}} = (3^5)^{\frac{1}{4}}$$

which means that

$$3^{\frac{5}{4}} = \sqrt[4]{3^5}$$

Since $3^5 = 243$, we could just write $3^{\frac{5}{4}} = \sqrt[4]{243}$.

Alternatively, we could express the fraction $\frac{5}{4}$ as $\frac{1}{4} \cdot 5$ instead. Using exponent rules, we get

$$3^{\frac{5}{4}} = \left(3^{\frac{1}{4}}\right)^5 = \left(\sqrt[4]{3}\right)^5$$

Here are more examples of exponents that are fractions and their equivalents:

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
5^x (using exponents)	5^0	$5^{\frac{1}{3}}$	$5^{\frac{2}{3}}$	5^1	$5^{\frac{4}{3}}$	$5^{\frac{5}{3}}$	5^2
5^x (equivalent expression)	1	$\sqrt[3]{5}$	$\sqrt[3]{5^2}$ or $\sqrt[3]{25}$	5	$\sqrt[3]{5^4}$ or $\sqrt[3]{625}$	$\sqrt[3]{5^5}$ or $\sqrt[3]{3125}$	25

Lesson 4 Practice Problems

1. Evaluate $8^{\frac{5}{3}}$.

2. Select **all** expressions that are equal to $64^{\frac{3}{2}}$.

    A. 96

    B. 8^3

    C. 512

    D. 4^2

    E. $\sqrt{64^3}$

    F. $\sqrt[3]{64}^2$

3. Write the expression $17^{\frac{4}{3}}$ using radicals.

4. An arithmetic sequence k starts 4, 13, ... Explain how you would calculate the value of the 5,000th term.

    (From Unit 1, Lesson 8.)

5. Select **all** items equivalent to $\sqrt{24}$.

A. the area of a square with side length 24 units

B. the side length of a square with area 24 square units

C. the positive number x, where $x \cdot x = 24$

D. the positive number y, where $y = 24 \cdot 24$

E. the edge length of a cube with volume 24 cubic units

F. the volume of a cube with edge length 24 units

(From Unit 3, Lesson 2.)

6. Which expression is equivalent to $23^{\frac{1}{2}}$?

A. $\frac{1}{23}$

B. $\frac{1}{\sqrt{23}}$

C. 11.5

D. $\sqrt{23}$

(From Unit 3, Lesson 3.)