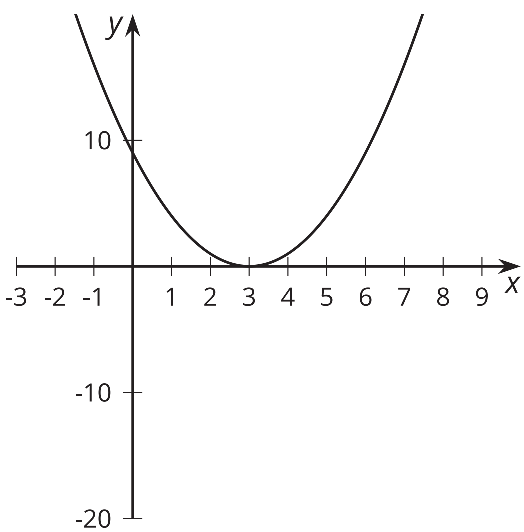
Lesson 10: Multiplicity

- Let's sketch some polynomial functions.

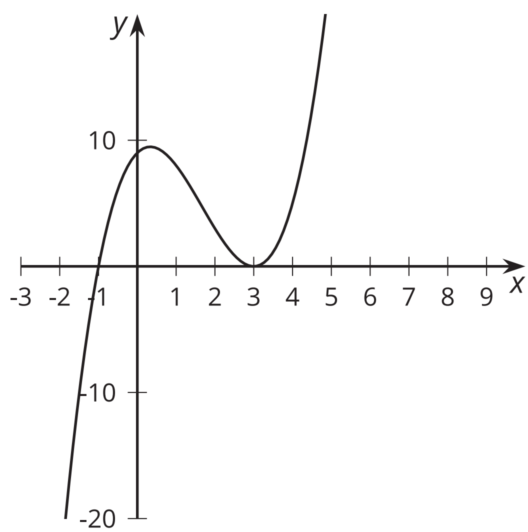
10.1: Notice and Wonder: Duplicate Factors

What do you notice? What do you wonder?

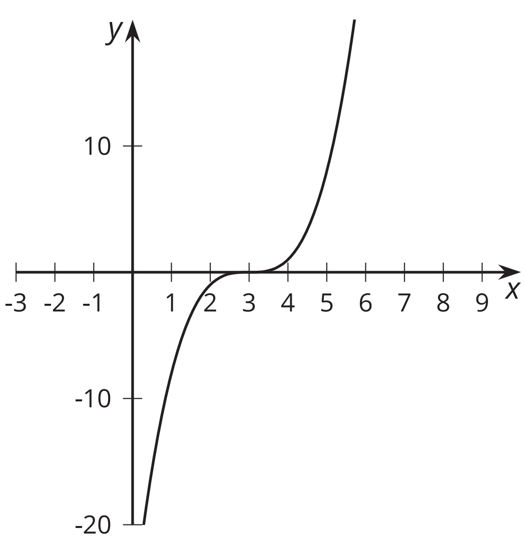
$y = (x-3)^2$



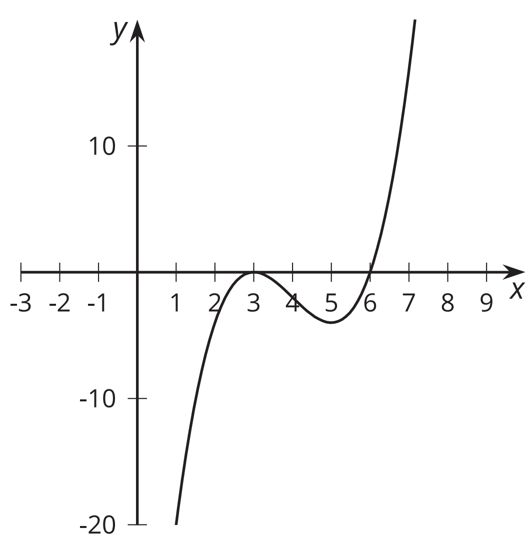
$y = (x+1)(x-3)^2$



$y = (x-3)^3$



$y = (x-6)(x-3)^2$



10.2: Sketching Polynomials

1. For polynomials A–F:
    a. Write the degree, all zeros, and complete the sentence about the end behavior.

    b. Sketch a possible graph.

    c. Check your sketch using graphing technology.

    Pause here for your teacher to check your work.

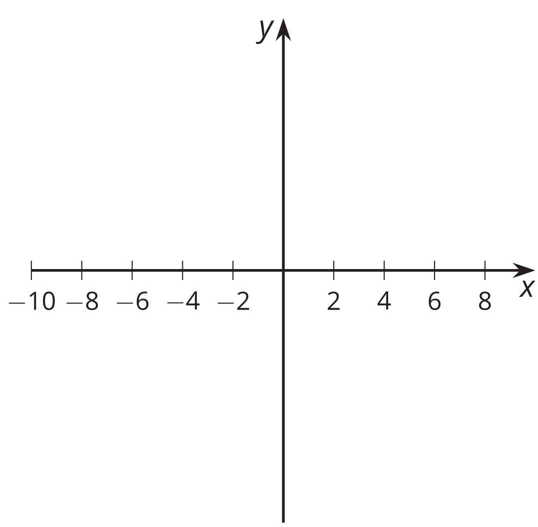
2. Create your own polynomial for your partner to figure out.
    a. Create a polynomial with degree greater than 2 and less than 8 and write the equation in the space given.

    b. Trade papers with a partner, then fill out the information about their polynomial and complete a sketch.

    c. Trade papers back. Check your partner's sketch using graphing technology.

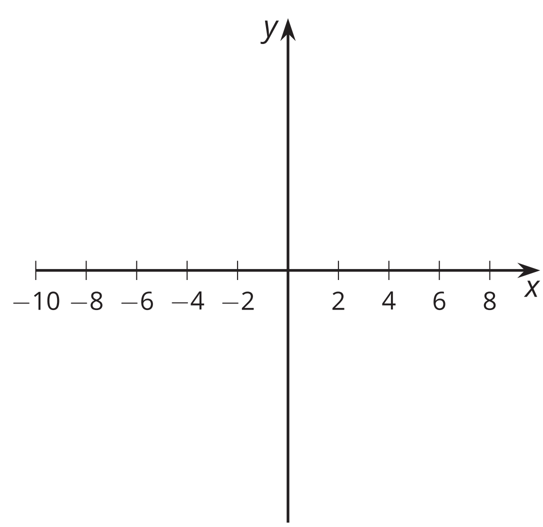
$A(x) = (x+2)(x-2)(x-8)$

Degree: Zeros:
End behavior: As x gets larger and larger in the negative direction,



$B(x) = -(x+2)(x-2)^2$

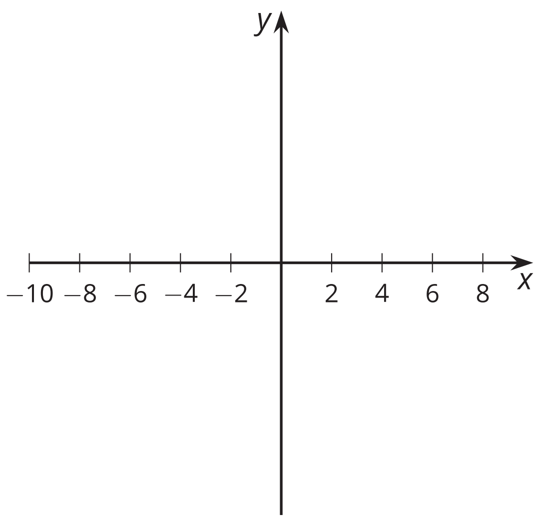
Degree: Zeros:
End behavior: As x gets larger and larger in the negative direction,



$C(x) = (x+6)(x+2)^2$

Degree: Zeros:

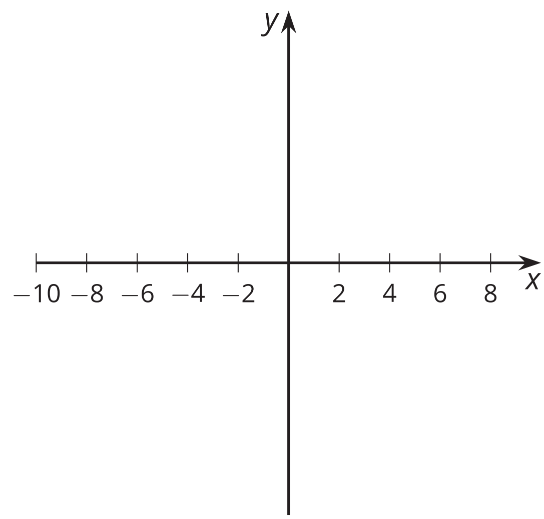
End behavior: As x gets larger and larger in the negative direction,



$D(x) = -(x+6)^2(x+2)$

Degree: Zeros:

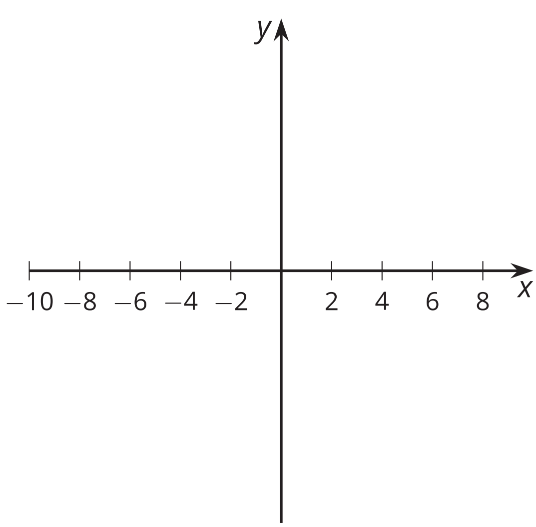
End behavior: As x gets larger and larger in the negative direction,



$E(x) = (x+4)(x-2)^3$

Degree: Zeros:

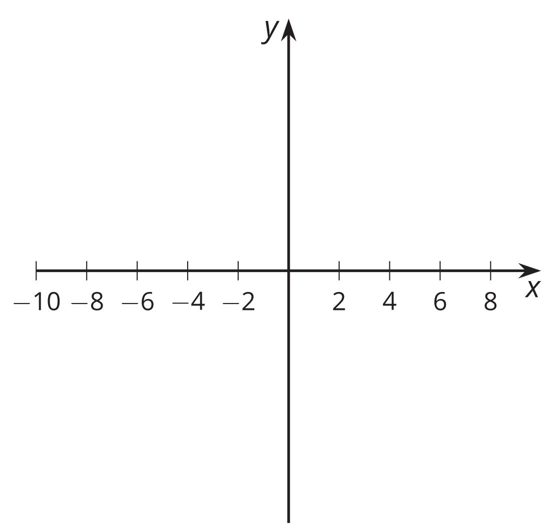
End behavior: As x gets larger and larger in the negative direction,



$F(x) = x^3(x+4)(x-3)^2$

Degree: Zeros:

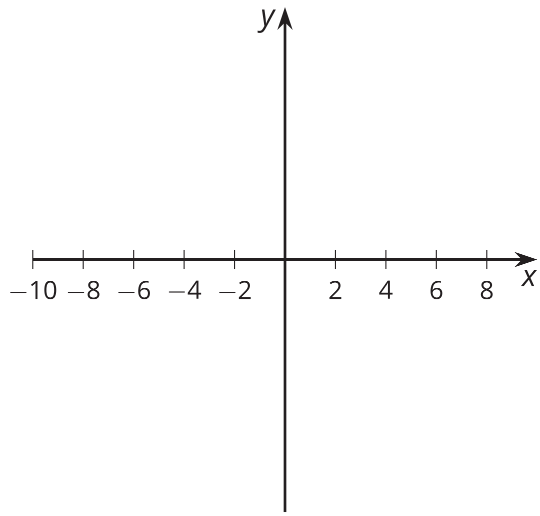
End behavior: As x gets larger and larger in the negative direction,



Your polynomial:

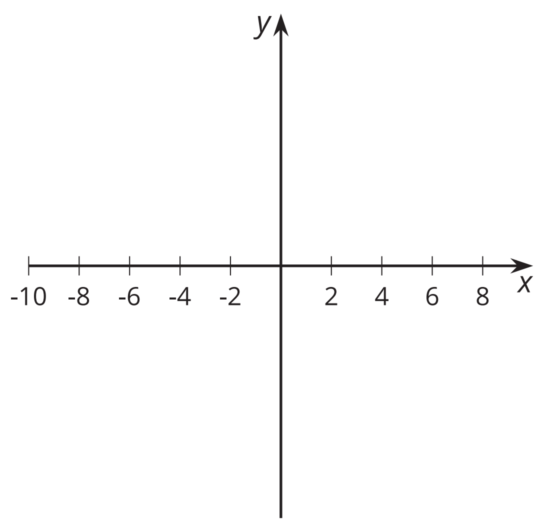
Degree: Zeros:

End behavior: As x gets larger and larger in the negative direction,



10.3: Using Knowledge of Zeros

1. Sketch a graph for a polynomial function $y = f(x)$ that has 3 different zeros and $f(x) \geq 0$ for all values of x.



2. What is the smallest degree the polynomial could have?

3. What is a possible equation for the polynomial? Use graphing technology to see if your equation matches your sketch.

Are you ready for more?

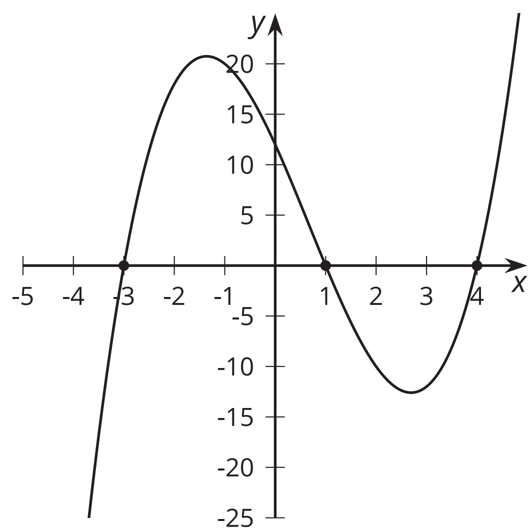
What is a possible equation of a polynomial function that has degree 5 but whose graph has exactly three horizontal intercepts and crosses the x-axis at all three intercepts? Explain why it is not possible to have a polynomial function that has degree 4 with this property.

Lesson 10 Summary

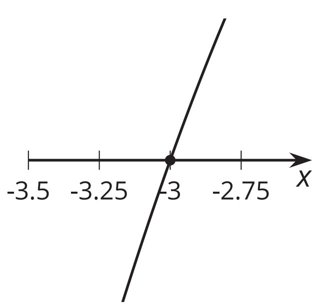
Earlier, we learned to identify the zeros of a polynomial function from the factored expression. These factors let us figure out the points where the graph of the polynomial intersects the horizontal axis. The number of times a factor is repeated also gives us important information: it tells us the shape of the graph at that point on the horizontal axis.

For example, $y=(x+3)(x-1)(x-4)$ has three factors with no duplicates. This results in a graph that looks a bit like a linear function near $x=-3$, $x=1$, and $x=4$ when we zoom in on each of those places.

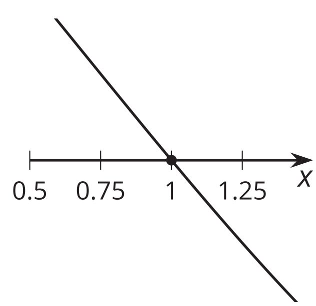
$y=(x+3)(x-1)(x-4)$



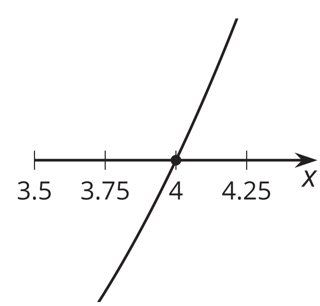
near $x=-3$



near $x=1$



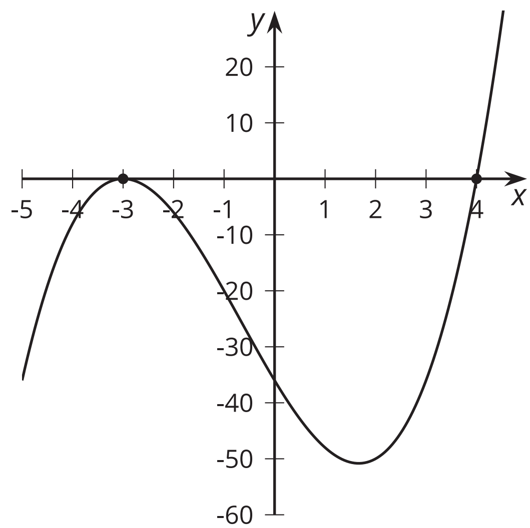
near $x=4$



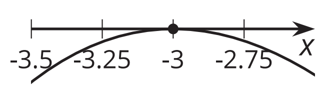
We say that each factor, $(x+3)$, $(x-1)$, and $(x-4)$, has a **multiplicity** of 1.

For $y = (x+3)^2(x-4)$, there are still three factors, but two of them are $(x+3)$. This results in a graph that looks a bit like a quadratic near $x = -3$ and a bit like a linear function near $x = 4$. We say that the factor $(x+3)$ has a multiplicity of 2 while the factor $(x-4)$ has a multiplicity of 1.

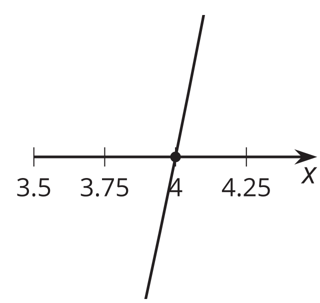
$y = (x+3)^2(x-4)$



near $x = -3$



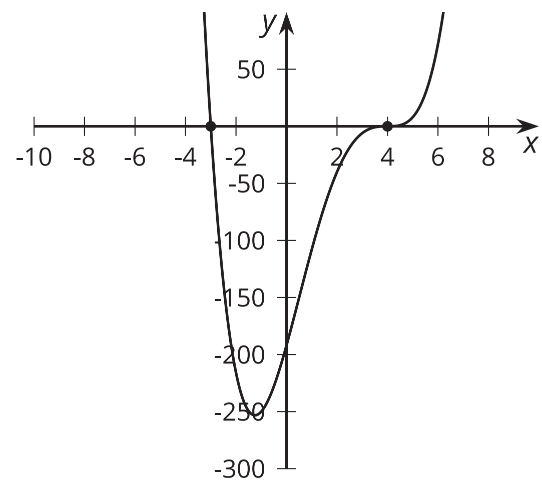
near $x = 4$



Combining what we know about factors, degree, end behavior, the sign of the leading coefficient, and multiplicity gives us the ability to sketch polynomials written in factored form.

For example, consider what the graph of $y = (x+3)(x-4)^3$ would look like. The factors help us identify that the function has zeros at -3 and 4. We also know that since $(x+3)$ has a multiplicity of 1 and $(x-4)$ has a multiplicity of 3, the graph looks a bit like a linear polynomial crossing the x-axis at -3 and a bit like a cubic polynomial crossing the x-axis at 4. Since this is a 4th degree polynomial with a positive leading coefficient, we know that as x gets larger and larger in either the negative or positive direction, y gets larger and larger in the positive direction.

$y = (x+3)(x-4)^3$



Glossary

- multiplicity

Lesson 10 Practice Problems

1. Draw a rough sketch of the graph of $g(x) = (x - 3)(x + 1)(7x - 2)$.

2. Draw a rough sketch of the graph of $f(x) = (x + 1)^2(x - 4)$.

3. *Technology required*. Predict the end behavior of each polynomial function, then check your prediction using technology.

   a. $A(x) = (x + 3)(x - 4)(3x - 7)(4x - 3)$

   b. $B(x) = (3 - x)^2(6 - x)$

   c. $C(x) = \text{-}(4 - 3x)(x^4)$

   d. $D(x) = (6 - x)^6$

4. Which term can be added to the polynomial expression $5x^7 - 6x^6 + 4x^4 - 4x^2$ to make it into a 10th degree polynomial?

   A. 10

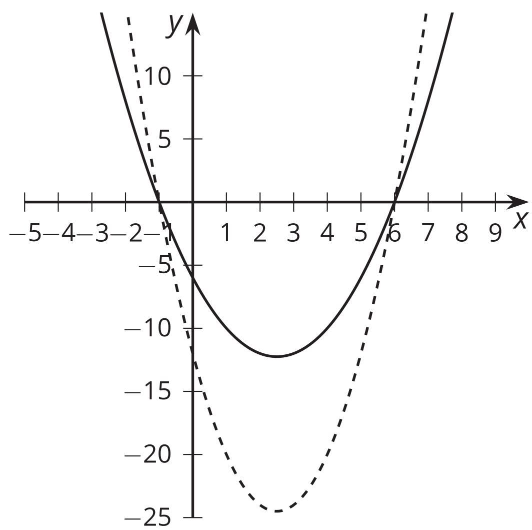
   B. $5x^3$

   C. $5x^7$

   D. x^{10}

   (From Unit 2, Lesson 3.)

5. $f(x) = (x + 1)(x - 6)$ and $g(x) = 2(x + 1)(x - 6)$. The graphs of each are shown.



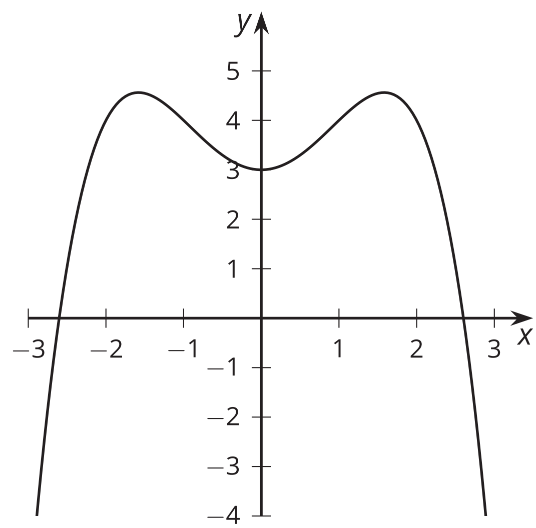
   a. Which graph represents which polynomial function? Explain how you know.

   (From Unit 2, Lesson 6.)

6. State the degree and end behavior of $f(x) = 8x^3 + 2x^4 - 5x^2 + 9$. Explain or show your reasoning.

(From Unit 2, Lesson 8.)

7. The graph of a polynomial function f is shown. Select **all** the true statements about the polynomial.



A. The degree of the polynomial is even.

B. The degree of the polynomial is odd.

C. The leading coefficient is positive.

D. The leading coefficient is negative.

E. The constant term of the polynomial is positive.

F. The constant term of the polynomial is negative.

(From Unit 2, Lesson 9.)

Lesson 11: Finding Intersections

- Let's think about two polynomials at once.

11.1: Math Talk: When f Meets g

Mentally identify a point where the graphs of the two functions intersect, if one exists.

$f(x) = x$ and $g(x) = 3$

$j(x) = (x + 3)(x - 3)$ and $k(x) = 0$

$m(x) = (x + 3)(x - 3)$ and $n(x) = (x - 3)$

$p(x) = (x + 5)(x - 5)$ and $q(x) = (x + 3)(x - 3)$

11.2: More Points of Intersection

For each pair of polynomials given, find all points of intersection of their graphs.

1. $c(x) = x^2 - 7$ and $d(x) = 2$

2. $f(x) = (x + 7)(x - 4)$ and $g(x) = x - 4$

3. $m(x) = (x + 7)(x - 4)$ and $n(x) = (2x + 5)(x - 4)$

4. $p(x) = (x + 1)(x - 8)$ and $q(x) = (x + 2)(x - 4)$

Are you ready for more?

Find all points of intersection of the graphs of the equations $p(x) = (2x + 3)(x - 5)$ and $q(x) = (x + 5)(x + 1)(x - 3)$. Use graphing technology to check your solutions.

11.3: Graphing to Find Points of Intersection

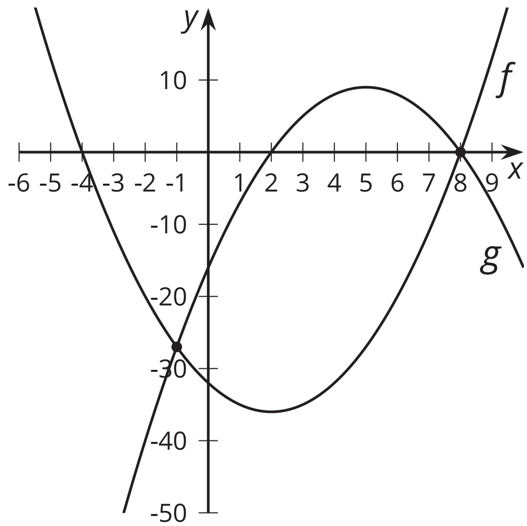
Consider the functions $p(x) = 5x^3 + 6x^2 + 4x$ and $q(x) = 5640$.

1. Use graphing technology to find a value of x that makes $p(x) = q(x)$ true.
2. For the x-value at the point of intersection, what can you say about the value of $5x^3 + 6x^2 + 4x - 5640$?
3. What does your answer suggest is a possible factor of $5x^3 + 6x^2 + 4x - 5640$?
4. a. Write your own polynomial $m(x)$ of degree 3 or higher.

   b. Use graphing technology to estimate the values of x that make $m(x) = q(x)$ true.

Lesson 11 Summary

When asked to find all values of x that make an equation like $(x+4)(x-8)=(2-x)(x-8)$ true, one way to consider the question is to ask where the graphs of the functions $f(x)=(x+4)(x-8)$ and $g(x)=(2-x)(x-8)$ intersect.



Since the coordinate of any point of intersection has the form $(a, f(a))=(a, g(a))$, these points must make $f(x)=g(x)$ true when $x=a$. In our example, we can tell from the graph that both $x=\text{-}1$ and $x=8$ are solutions to the original equation.

We can also use algebra to identify solutions to $(x+4)(x-8)=(2-x)(x-8)$ by rearranging and then recognizing that both parts have a factor of $(x-8)$ in common:

$$\begin{aligned}(x+4)(x-8)&=(2-x)(x-8)\\(x+4)(x-8)-(2-x)(x-8)&=0\\(x-8)(x+4-2+x)&=0\\(x-8)(2x+2)&=0\\x&=8,\text{-}1\end{aligned}$$

For polynomials created to model specific situations that have a more messy structure, solving without using technology can be challenging, especially because the graphs of two polynomials can intersect at multiple points because of the way they curve. Fortunately, this type of solving challenge is one that computer algebra systems are usually very good at, leaving the interpretation of the solution up to humans.

Lesson 11 Practice Problems

1. What are the points of intersection between the graphs of the functions $f(x) = x^2(x+1)$ and $g(x) = x+1$?

2. Select **all** the points of intersection between the graphs of the functions $f(x) = (x+5)(x-2)$ and $g(x) = (2x+1)(x-2)$.

   A. $(-5, 0)$

   B. $(-\frac{1}{2}, 0)$

   C. $(-2, -12)$

   D. $(2, 0)$

   E. $(4, 18)$

   F. $(5, 30)$

3. What are the solutions to the equation $(x-3)(x+5) = -15$?

4. What are the x-intercepts of the graph of $y = (5x+7)(2x-1)(x-4)$?

   A. $-\frac{7}{5}, -\frac{1}{2}, 4$

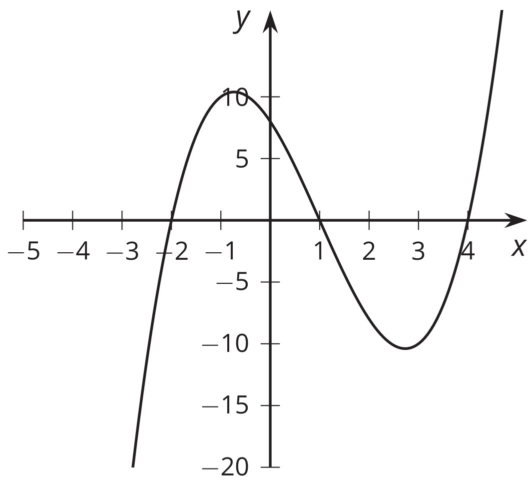
   B. $\frac{5}{7}, \frac{1}{2}, 4$

   C. $-\frac{7}{5}, \frac{1}{2}, 4$

   D. $\frac{5}{7}, 2, 4$

   (From Unit 2, Lesson 5.)

5. Which polynomial function's graph is shown here?



A. $f(x) = (x+1)(x+2)(x+4)$

B. $f(x) = (x+1)(x-2)(x+4)$

C. $f(x) = (x-1)(x+2)(x-4)$

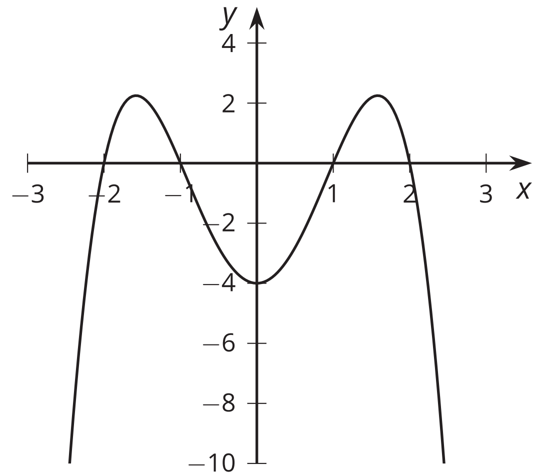
D. $f(x) = (x-1)(x-2)(x-4)$

(From Unit 2, Lesson 7.)

6. Draw a rough sketch of the graph of $g(x) = -x^2(x+2)$.

(From Unit 2, Lesson 10.)

7. The graph of a polynomial function f is shown.



a. Is the degree of the polynomial odd or even? Explain how you know.

b. What is the constant term of the polynomial?

(From Unit 2, Lesson 9.)