Lesson 7: Interpreting and Using Exponential Functions

- Let's explore the ages of ancient things.

7.1: Halving and Doubling

1. A colony of microbes doubles in population every 6 hours. Explain why we could say that the population grows by a factor of $\sqrt[6]{2}$ every hour.

2. A bacteria population decreases by a factor of $\frac{1}{2}$ every 4 hours. Explain why we could also say that the population decays by a factor of $\sqrt[4]{\frac{1}{2}}$ every hour.

7.2: Radiocarbon Dating

Carbon-14 is used to find the age of certain artifacts and fossils. It has a half-life of 5,730 years, so if an object has carbon-14, it loses half of it every 5,730 years.

1. At a certain point in time, a fossil had 3 picograms (a trillionth of a gram) of carbon-14. Complete the table with the missing mass of carbon-14 and years.

number of years after fossil had 3 picograms of carbon-14	mass of carbon-14 in picograms
0	3
1,910	
5,730	
	0.75

2. A scientist uses the expression $(2.5) \cdot \left(\frac{1}{2}\right)^{\frac{t}{5,730}}$ to model the number of picograms of carbon-14 remaining in a different fossil t years after 20,000 BC.
    a. What do the 2.5, $\frac{1}{2}$, and 5,730 mean in this situation?

    b. Would more or less than 0.1 picogram of carbon-14 remain in this fossil today? Explain how you know.

7.3: Old Manuscripts

The half-life of carbon-14 is about 5,730 years.

1. Pythagoras lived between 600 BCE and 500 BCE. Explain why the age of a papyrus from the time of Pythagoras is about half of a carbon-14 half-life.

2. Someone claims they have a papyrus scroll written by Pythagoras. Testing shows the scroll has 85% of its original amount of carbon-14 remaining. Explain why the scroll is likely a fake.

Are you ready for more?

A copy of the Gutenberg Bible was made around 1450. Would more or less than 90% of the carbon-14 remain in the paper today? Explain how you know.

Lesson 7 Summary

Some substances change over time through a process called radioactive decay, and their rate of decay can be measured or estimated. Let’s take sodium-22 as an example.

Suppose a scientist finds 4 nanograms of sodium-22 in a sample of an artifact. (One nanogram is 1 billionth, or 10^{-9}, of a gram.) Approximately every 3 years, half of the sodium-22 decays. We can represent this change with a table.

number of years after first being measured	mass of sodium-22 in nanograms
0	4
3	2
6	1
9	0.5

This can also be represented by an equation. If the function f gives the number of nanograms of sodium remaining after t years then

$$f(t) = 4 \cdot \left(\frac{1}{2}\right)^{\frac{t}{3}}$$

The 4 represents the number of nanograms in the sample when it was first measured, while the $\frac{1}{2}$ and 3 show that the amount of sodium is cut in half every 3 years, because if you increase t by 3, you increase the exponent by 1.

How much of the sodium remains after one year? Using the equation, we find $f(1) = 4 \cdot \left(\frac{1}{2}\right)^{\frac{1}{3}}$. This is about 3.2 nanograms.

About how many years after the first measurement will there be about 0.015 nanogram of sodium-22? One way to find out is by extending the table and multiplying the mass of sodium-22 by $\frac{1}{2}$ each time. If we multiply 0.5 nanogram (the mass of sodium-22 9 years after first being measured) by $\frac{1}{2}$ five more times, the mass is about 0.016 nanogram. For sodium-22, five half-lives means 15 years, so 24 years after the initial measurement, the amount of sodium-22 will be about 0.015 nanogram.

Archaeologists and scientists use exponential functions to help estimate the ages of ancient things.

Lesson 7 Practice Problems

1. The half-life of carbon-14 is about 5,730 years. A fossil had 6 picograms of carbon-14 at one point in time. (A picogram is a trillionth of a gram or 1×10^{-12} gram.) Which expression describes the amount of carbon-14, in picograms, t years after it was measured to be 6 picograms.

   A. $6 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5,730}}$

   B. $6 \cdot \left(\frac{1}{2}\right)^{5,730t}$

   C. $6 \cdot (5,730)^{\frac{1}{2}t}$

   D. $\frac{1}{2} \cdot (6)^{\frac{t}{5,730}}$

2. The half-life of carbon-14 is about 5,730 years. A tree fossil was estimated to have about 4.2 picograms of carbon-14 when it died. (A picogram is a trillionth of a gram.) The fossil now has about 0.5 picogram of carbon-14. About how many years ago did the tree die? Show your reasoning.

3. Nickel-63 is a radioactive substance with a half-life of about 100 years. An artifact had 9.8 milligrams of nickel-63 when it was first measured. Write an equation to represent the mass of nickel-63, in milligrams, as a function of:

   a. t, time in years

   b. d, time in days

4. Tyler says that the function $f(x) = 5^x$ is exponential and so it grows by equal factors over equal intervals. He says that factor must be $\sqrt[10]{5}$ for an interval of $\frac{1}{10}$ because ten of those intervals makes an interval of length 1. Do you agree with Tyler? Explain your reasoning.

(From Unit 4, Lesson 5.)

5. The population in a city is modeled by the equation $p(d) = 100{,}000 \cdot (1 + 0.3)^d$, where d is the number of decades since 1970.

   a. What do the 0.3 and 100,000 mean in this situation?

   b. Write an equation for the function f to represent the population y years after 1970. Show your reasoning.

   c. Write an equation for the function g to represent the population c centuries after 1970. Show your reasoning.

(From Unit 4, Lesson 6.)

6. The function f is exponential. Its graph contains the points $(0, 5)$ and $(1.5, 10)$.

   a. Find $f(3)$. Explain your reasoning.

   b. Use the value of $f(3)$ to find $f(1)$. Explain your reasoning.

   c. What is an equation that defines f?

(From Unit 4, Lesson 6.)

7. Select **all** expressions that are equal to $8^{\frac{2}{3}}$.

   A. $\sqrt[3]{8^2}$

   B. $\sqrt[3]{8}^2$

   C. $\sqrt{8^3}$

   D. 2^2

   E. 2^3

   F. 4

(From Unit 3, Lesson 4.)