Lesson 21: Rational Equations (Part 2)

- Let's write and solve some more rational equations.

21.1: Math Talk: Adding Rationals

Solve each equation mentally:

$$\frac{x}{2} = \frac{3}{4}$$

$$\frac{3}{x} = \frac{1}{6}$$

$$\frac{1}{4} = \frac{1}{x^2}$$

$$\frac{2}{x} = \frac{x}{8}$$

21.2: A Rational River

Noah likes to go for boat rides along a river with his family. In still water, the boat travels about 8 kilometers per hour. In the river, it takes them the same amount of time t to go upstream 5 kilometers as it does to travel downstream 10 kilometers.

1. If the speed of the river is r, write an expression for the time it takes to travel 5 kilometers upstream and an expression for the time it takes to travel 10 kilometers downstream.

2. Use your expressions to calculate the speed of the river. Explain or show your reasoning.

21.3: Rational Resistance

Circuits in parallel follow this law: The inverse of the total resistance is the sum of the inverses of each individual resistance. We can write this as: $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$ where there are n parallel circuits and R_T is the total resistance. Resistance is measured in ohms.

1. Two circuits are placed in parallel. The first circuit has a resistance of 40 ohms and the second circuit has a resistance of 60 ohms. What is the total resistance of the two circuits?

2. Two circuits are placed in parallel. The second circuit has a resistance of 150 ohms more than the first. Write an equation for this situation showing the relationships between R_T and the resistance R of the first circuit.

3. For this circuit, Clare wants to use graphs to estimate the resistance of the first circuit R if R_T is 85 ohms. Describe how she could use a graph to determine the value of R and then follow your instructions to find R.

Are you ready for more?

Two circuits with resistances of 40 ohms and 60 ohms have a combined resistance of 24 ohms when connected in parallel. If we had used two circuits that each had a resistance of 48 ohms, they would have had that same combined resistance. 48 is called the harmonic mean of 40 and 60. A more familiar way to find the mean of two numbers is to add them up and divide by 2. This is the arithmetic mean. Here is how each kind of mean is calculated:

Harmonic mean of a and b:

$$\frac{2ab}{a+b}$$

Arithmetic mean of a and b:

$$\frac{a+b}{2}$$

The harmonic mean of 40 and 60 was 48, and their arithmetic mean is (40+60)/2=50. Experiment with other pairs of numbers. What can you conclude about the relationship between the harmonic mean and arithmetic mean?

Lesson 21 Summary

A boat travels about 6 kilometers per hour in still water. If the boat is on a river that flows at a constant speed of r kilometers per hour, it can travel at a speed of $6 + r$ kilometers per hour downstream and $6 - r$ kilometers per hour upstream. (And if the river current is the same speed as the boat, the boat wouldn't be able to travel upstream at all!)

On one particular river, the boat can travel 4 kilometers upstream in the same amount of time it takes to travel 12 kilometers downstream. Since time is equal to distance divided by speed, we can express the travel time as either $\frac{12}{6+r}$ hours or $\frac{4}{6-r}$ hours. If we don't know the travel time, we can make an equation using the fact that these two expressions are equal to one another, and figure out the speed of the river.

$$\begin{aligned} \frac{12}{6+r} &= \frac{4}{6-r} \\ \frac{12}{6+r} \boldsymbol{\cdot} (6+r)(6-r) &= \frac{4}{6-r} \boldsymbol{\cdot} (6+r)(6-r) \\ 12(6-r) &= 4(6+r) \\ 72 - 12r &= 24 + 4r \\ 48 &= 16r \\ 3 &= r \end{aligned}$$

Substituting this value into the original expressions, we have $\frac{12}{6+3} = \frac{4}{3}$ and $\frac{4}{6-3} = \frac{4}{3}$, so these two expressions are equal when $r = 3$. This means that when the water flow in the river is about 3 kilometers per hour, it takes the boat 1 hour and 20 minutes to go 4 kilometers upstream and 1 hour and 20 minutes to go 12 kilometers downstream.

Even though we started out with a rational expression on each side of the equation, multiplying each side by the product of the denominators, $(6 + r)(6 - r)$, resulted in an equation similar to ones we have solved before. Multiplying to get an equation with no variables in denominators is sometimes called "clearing the denominators."

Lesson 21 Practice Problems

1. Solve $x - 1 = \dfrac{x^2 - 4x + 3}{x + 2}$ for x.

2. Solve $\frac{4}{4-x} = \frac{5}{4+x}$ for x.

3. Show that the equation $\frac{1}{60} = \frac{2x+50}{x(x+50)}$ is equivalent to $x^2 - 70x - 3{,}000 = 0$ for all values of x not equal to 0 or -50. Explain each step as you rewrite the original equation.

4. Kiran jogs at a speed of 6 miles per hour when there are no hills. He plans to jog up a mountain road, which will cause his speed to decrease by r miles per hour. Which expression represents the time, t, in hours it will take him to jog 8 miles up the mountain road?

   A. $t = \frac{8-r}{6}$

   B. $t = \frac{8}{6+r}$

   C. $t = \frac{6+r}{8}$

   D. $t = \frac{8}{6-r}$

5. The rational function $g(x) = \frac{x+10}{x}$ can be rewritten in the form $g(x) = c + \frac{r}{x}$, where c and r are constants. Which expression is the result?

   A. $g(x) = x + \frac{10}{x}$

   B. $g(x) = 1 + \frac{10}{x}$

   C. $g(x) = x - \frac{10}{x+10}$

   D. $g(x) = 1 - \frac{1}{x+10}$

   (From Unit 2, Lesson 18.)

6. For each equation below, find the value(s) of x that make it true.

   a. $10 = \frac{1+7x}{7+x}$

   b. $0.2 = \frac{6+2x}{12+x}$

   c. $0.8 = \frac{x}{0.5+x}$

   d. $3.5 = \frac{4+2x}{0.5-x}$

   (From Unit 2, Lesson 20.)

7. A softball player has had 8 base hits out of 25 at bats for a current batting average of $\frac{8}{25} = .320$.

   How many consecutive base hits does she need if she wants to raise her batting average to .400? Explain or show your reasoning.

   (From Unit 2, Lesson 20.)

Lesson 22: Solving Rational Equations

- Let's think about how to solve rational equations strategically.

22.1: Notice and Wonder: Thoughtful Multiplication

What do you notice? What do you wonder?

$$\frac{3}{x(x-2)} = \frac{2x+1}{x-2}$$
$$\frac{3}{x(x-2)} \cdot x(x-2) = \frac{2x+1}{x-2} \cdot x(x-2)$$
$$3 = 2x^2 + x$$
$$0 = 2x^2 + x - 3$$

22.2: Rational Solving

Jada is working to find values of x that make this equation true:

$$\frac{5x+5}{x+1} = \frac{5}{x}$$

She says, "If I multiply both sides by $x(x+1)$, I find that the solutions are $x = 1$ and $x = -1$, but when I substitute in $x = -1$, the equation does not make any sense."

1. Is Jada's work correct? Explain or show your reasoning.

2. Why does Jada's method produce an x value that does not solve the equation?

Are you ready for more?

1. What are the solutions to $x^2 = 1$?

2. What are the solutions to $\frac{x^2}{x-1} = \frac{1}{x-1}$?

3. How can you solve $\frac{x^2}{x-1} = \frac{1}{x-1}$ by inspection?

4. How does the denominator influence the solution(s) to $\frac{x^2}{x-1} = \frac{1}{x-1}$?

22.3: More Rational Solving

1. Here are a lot of equations. For each one, use what you know about division to identify values of x that cannot be solutions to the equation.
   a. $\frac{x^2+x-6}{x-2} = 5$
   b. $\frac{2x+1}{x} = \frac{1}{x-2}$
   c. $\frac{10}{x+8} = \frac{5}{x-8}$
   d. $\frac{x^2+x+1}{13} = \frac{2}{x-1}$
   e. $\frac{x+1}{4x} = \frac{x-1}{3x}$
   f. $\frac{1}{x} = \frac{1}{x(x+1)}$
   g. $\frac{x+2}{x} = \frac{3}{x-2}$
   h. $\frac{1}{x-3} = \frac{1}{x(x-3)}$
   i. $\frac{(x+1)(x+2)}{x+1} = \frac{x+2}{x+1}$
2. Without solving, identify three of the equations that you think would be least difficult to solve and three that you think would be most difficult to solve. Be prepared to explain your reasoning.

3. Choose three equations to solve. At least one should be from your “least difficult” list and one should be from your “most difficult” list.

Lesson 22 Summary

Consider the equation $\frac{x+2}{x(x+1)} = \frac{2}{(x+1)(x-1)}$. We could solve this equation for x by multiplying each expression by $x(x+1)(x-1)$ to get an equation with no variables in denominators, and then rearranging it into an expression that equals 0. Here is what that looks like:

$$\begin{aligned}
\frac{x+2}{x(x+1)} \cdot x(x+1)(x-1) &= \frac{2}{(x+1)(x-1)} \cdot x(x+1)(x-1) \\
(x+2)(x-1) &= 2x \\
x^2 + x - 2 &= 2x \\
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0
\end{aligned}$$

The last equation, $(x-2)(x+1) = 0$, leads us to believe that the original equation has two solutions: $x = 2$ and $x = \text{-}1$. Substituting $x = 2$ into the original equation, we get $\frac{2+2}{2(2+1)} = \frac{2}{(2+1)(2-1)}$, which is true since each side is equal to $\frac{2}{3}$. But, substituting $x = \text{-}1$ into the original equation, we get $\frac{\text{-}1+2}{\text{-}1(\text{-}1+1)} = \frac{2}{(\text{-}1+1)(\text{-}1-1)}$, which isn't a valid equation since division by 0 is not allowed. This means $x = \text{-}1$ isn't a solution, so what happened to make us think that it was?

Let's consider the simpler equation $x - 5 = 0$. This equation has one solution, $x = 5$. But if we multiply each side by $(x-1)$ the result is a new equation, $(x-1)(x-5) = 0$, which has solutions 5 and 1. The 1 is a solution to the new equation because when $x = 1$, $x - 1 = 0$. But if we substitute 1 for x into the original equation, we get $1 - 5 = \text{-}4 = 0$, which is not a valid equation, so 1 is not a solution to the original equation. Because we multiplied each side of the original equation by an expression that has the value 0 when $x = 1$, the two sides $x - 5$ and 0 that were unequal at that specific x-value are now equal. For this example, $x = 1$ is sometimes called an *extraneous solution*.

In the original example, $x = \text{-}1$ is the extraneous solution. While $x = \text{-}1$ is a solution to the equation we wrote after we multiplied the original equation by $x(x+1)(x-1)$ on each side, it is not a solution to the original equation since they are not equivalent. It should be noted that even though we multiplied by x, $(x+1)$, and $(x-1)$, only one extraneous solution was added. This shows that multiplying by an expression that can equal zero does not always cause an extraneous solution. So how do we tell if a solution is extraneous or not? We substitute it into the original equation and make sure the result is a valid equation.

Lesson 22 Practice Problems

1. Identify all values of x that make the equation true.

    a. $\frac{2x+1}{x} = \frac{1}{x-2}$

    b. $\frac{1}{x+2} = \frac{2}{x-1}$

    c. $\frac{x+3}{1-x} = \frac{x+1}{x+2}$

    d. $\frac{x+2}{x+8} = \frac{1}{x+2}$

2. Kiran is solving $\frac{2x-3}{x-1} = \frac{2}{x(x-1)}$ for x, and he uses these steps:

$$\frac{2x-3}{x-1} = \frac{2}{x(x-1)}$$
$$(x-1)\left(\frac{2x-3}{x-1}\right) = x(x-1)\left(\frac{2}{x(x-1)}\right)$$
$$2x-3 = 2$$
$$2x = 5$$
$$x = 2.5$$

He checks his answer and finds that it isn't a solution to the original equation, so he writes "no solutions." Unfortunately, Kiran made a mistake while solving. Find his error and calculate the actual solution(s).

3. Identify all values of x that make the equation true.

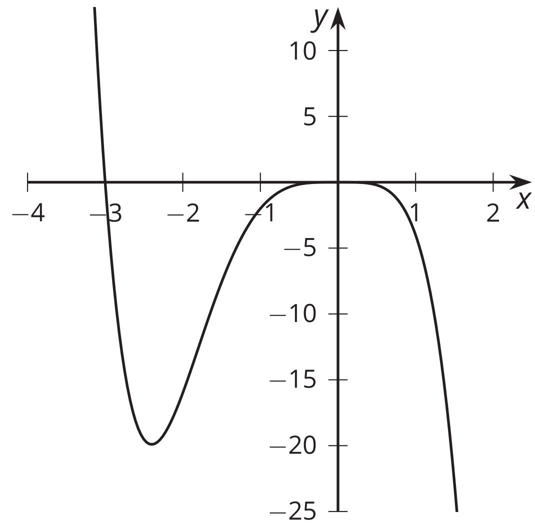
a. $x = \frac{25}{x}$

b. $x + 2 = \frac{6x-3}{x}$

c. $\frac{x}{x^2} = \frac{3}{x}$

d. $\frac{6x^2+18x}{2x^3} = \frac{5}{x}$

4. Is this the graph of $g(x) = -x^4(x+3)$ or $h(x) = x^4(x+3)$? Explain how you know.



(From Unit 2, Lesson 10.)

5. Rewrite the rational function $g(x) = \frac{x-9}{x}$ in the form $g(x) = c + \frac{r}{x}$, where c and r are constants.

(From Unit 2, Lesson 18.)

6. Elena has a boat that would go 9 miles per hour in still water. She travels downstream for a certain distance and then back upstream to where she started. Elena notices that it takes her 4 hours to travel upstream and 2 hours to travel downstream. The river's speed is r miles per hour. Write an expression that will help her solve for r.

(From Unit 2, Lesson 21.)