Lesson 19: End Behavior of Rational Functions

- Let’s explore the end behavior of rational functions.

19.1: Different Divisions, Revisited

Complete all three representations of the polynomial division following the forms of the integer division.

$$\begin{array}{r} 252 \\ 11\,\overline{)\,2775} \\ \underline{2200} \\ 575 \\ \underline{550} \\ 25 \\ \underline{22} \\ 3 \end{array}$$

$$\begin{array}{r} 2x^2 \\ x+1\,\overline{)\,2x^3+7x^2+7x+5} \end{array}$$

$2775 = 11(252) + 3$

$2x^3 + 7x^2 + 7x + 5 =$

$\frac{2775}{11} = 252 + \frac{3}{11}$

$\frac{2x^3 + 7x^2 + 7x + 5}{x+1} =$

19.2: Combined Fuel Economy

In 2000, the Environmental Protection Agency (EPA) reported a combined fuel efficiency for cars that assumes 55% city driving and 45% highway driving. The expression for the combined fuel efficiency of a car that gets x mpg in the city and h mpg on the highway can be written as $\frac{100xh}{55x+45h}$.

1. Several conventional cars have a fuel economy for highway driving that is about 10 mpg higher than for city driving. That is, $h = x + 10$. Write a function f that represents the combined fuel efficiency for cars like these in terms of x.

2. Rewrite f in the form $q(x) + \frac{r(x)}{b(x)}$ where $q(x)$, $r(x)$, and $b(x)$ are polynomials.

19.3: Exploring End Behavior

function	degree of num.	degree of den.	rewritten in the form of $q(x) + \frac{r(x)}{b(x)}$	end behavior
$g(x) = -\frac{5}{x+2}$				
$h(x) = \frac{7x-5}{x+2}$				
$j(x) = \frac{3x^2+7x-5}{x+2}$				
$k(x) = \frac{2x^3+3x^2+7x-5}{x+2}$				
$m(x) = \frac{x+2}{2x^3+3x^2+7x-5}$				

1. Complete the table to explore the end behavior for rational functions.

2. What do you notice about the end behavior of different types of rational functions?

Are you ready for more?

1. Graph $y = j(x)$ and the line it approaches.

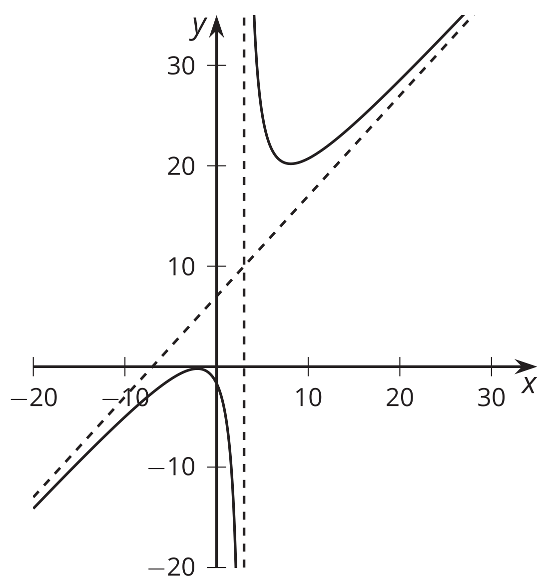
2. Under what conditions would the end behavior of the graph of a rational function approach a line that is not horizontal?

3. Create a rational function that approaches the line $y = 2x - 3$ as x gets larger and larger in either the positive or negative direction.

Lesson 19 Summary

In earlier lessons, we saw rational functions whose end behavior could be described by a horizontal asymptote. For example, we can rewrite functions like $d(x) = \frac{x+4}{x}$ as $d(x) = 1 + \frac{4}{x}$ to see more clearly that as x gets larger and larger in either the positive or negative direction, the value of $\frac{4}{x}$ gets closer and closer to 0, which means the value of $d(x)$ gets closer to 1. We can use similar thinking to understand rational functions that do not have horizontal asymptotes.

For example, consider $f(x) = \frac{x^2+4x+5}{x-3}$. Using division, the expression can be rewritten as $f(x) = x + 7 + \frac{26}{x-3}$. As x gets larger and larger in either the positive or negative direction, the value of the term $\frac{26}{x-3}$ gets closer and closer to 0, which means the value of $d(x)$ gets closer to the value of $x + 7$. This means that the end behavior of f can be described by the line $y = x + 7$. Here is a graph of $y = f(x)$, the line $y = x + 7$, and the vertical asymptote of the function at $x = 3$:



Lesson 19 Practice Problems

1. The function $f(x) = \frac{5x+2}{x-3}$ can be rewritten in the form $f(x) = 5 + \frac{17}{x-3}$. What is the end behavior of $y = f(x)$?

2. Rewrite the rational function $g(x) = \frac{x^2+7x-12}{x+2}$ in the form $g(x) = p(x) + \frac{r}{x+2}$, where $p(x)$ is a polynomial and r is an integer.

3. Match each polynomial with its end behavior as x gets larger and larger in the positive and negative directions. (Note: Some of the answer choices are not used and some answer choices are used more than once.)

A. $p(x) = \dfrac{3}{x-1}$

B. $q(x) = \dfrac{2x}{x-1}$

C. $r(x) = \dfrac{2x+3}{x-1}$

D. $s(x) = \dfrac{2x^2+x+3}{x-1}$

E. $t(x) = \dfrac{x^3}{x-1}$

1. The graph approaches $y = 2$.
2. The graph approaches $y = 3$.
3. The graph approaches $y = 2x + 3$.
4. The graph approaches $y = x^2 + x + 1$.
5. The graph approaches $y = 0$.

4. Let the function P be defined by $P(x) = x^3 + 2x^2 - 13x + 10$. Mai divides $P(x)$ by $x + 5$ and gets:

$$
\begin{array}{r}
x^2 - 3x + 2 \\
x + 5 \overline{\smash{\big)}\, x^3 + 2x^2 - 13x + 10} \\
\underline{-x^3 - 5x^2} \\
-3x^2 - 13x \\
\underline{3x^2 + 15x} \\
2x + 10 \\
\underline{-2x - 10} \\
0
\end{array}
$$

How could we tell by looking at the remainder that $(x + 5)$ is a factor?

(From Unit 2, Lesson 13.)

5. For the polynomial function $f(x) = x^4 + 3x^3 - x^2 - 3x$ we have $f(\text{-}3) = 0$, $f(\text{-}2) = \text{-}6$, $f(\text{-}1) = 0$, $f(0) = 0$, $f(1) = 0$, $f(2) = 30$, $f(3) = 144$. Rewrite $f(x)$ as a product of linear factors.

(From Unit 2, Lesson 15.)

6. There are many cones with a volume of 60π cubic inches. The height $h(r)$ in inches of one of these cones is a function of its radius r in inches where $h(r) = \frac{180}{r^2}$.

    a. What is the height of one of these cones if its radius is 2 inches?

    b. What is the height of one of these cones if its radius is 3 inches?

    c. What is the height of one of these cones if its radius is 6 inches?

(From Unit 2, Lesson 16.)

7. A cylindrical can needs to have a volume of 10 cubic inches. There needs to be a label around the side of the can. The function $S(r) = \frac{20}{r}$ gives the area of the label in square inches where r is the radius of the can in inches.

    a. As r gets closer and closer to 0, what does the behavior of the function tell you about the situation?

    b. As r gets larger and larger, what does the end behavior of the function tell you about the situation?

(From Unit 2, Lesson 17.)

8. Match each rational function with a description of its end behavior as x gets larger and larger.

A. $9x$

B. $\frac{9}{x}$

C. $\frac{99x}{x}$

D. $\frac{99+x}{x}$

E. $\frac{99x+9}{x}$

F. $\frac{99+9x}{x}$

1. The value of the expression gets closer and closer to 0.
2. The value of the expression gets closer and closer to 1.
3. The value of the expression gets closer and closer to 9.
4. The value of the expression is 99.
5. The value of the expression gets larger and larger in the positive direction.
6. The value of the expression gets larger and larger in the negative direction.

(From Unit 2, Lesson 18.)