Lesson 17: Completing the Square and Complex Solutions

- Let’s find complex solutions to quadratic equations by completing the square.

17.1: Creating Quadratic Equations

Match each equation in standard form to its factored form and its solutions.

1. $x^2 - 25 = 0$
2. $x^2 - 5 = 0$
3. $x^2 + 25 = 0$

- $(x - 5i)(x + 5i) = 0$
- $(x - 5)(x + 5) = 0$
- $(x - \sqrt{5})(x + \sqrt{5}) = 0$

- $\sqrt{5}$, -$\sqrt{5}$
- 5, -5
- $5i$, -$5i$

17.2: Sometimes the Solutions Aren't Real Numbers

What are the solutions to these equations?

1. $(x - 5)^2 = 0$
2. $(x - 5)^2 = 1$
3. $(x - 5)^2 = -1$

17.3: Finding Complex Solutions

Solve these equations by completing the square.

1. $x^2 - 8x + 13 = 0$

2. $x^2 - 8x + 19 = 0$

Are you ready for more?

For which values of a does the equation $x^2 - 8x + a = 0$ have two real solutions? One real solution? No real solutions? Explain your reasoning.

17.4: Can You See the Solutions on a Graph?

1. How many real solutions does each equation have? How many non-real solutions?
    a. $x^2 - 8x + 13 = 0$

    b. $x^2 - 8x + 16 = 0$

    c. $x^2 - 8x + 19 = 0$

2. How do the graphs of these functions help us answer the previous question?
    a. $f(x) = x^2 - 8x + 13$

    b. $g(x) = x^2 - 8x + 16$

    c. $h(x) = x^2 - 8x + 19$

Lesson 17 Summary

Sometimes quadratic equations have real solutions, and sometimes they do not. Here is a quadratic equation with x^2 equal to a negative number (assume k is positive):

$$x^2 = -k$$

This equation will have imaginary solutions $i\sqrt{k}$ and $-i\sqrt{k}$. By similar reasoning, an equation of the form:

$$(x-h)^2 = -k$$

will have non-real solutions if k is positive. In this case, the solutions are $h + i\sqrt{k}$ and $h - i\sqrt{k}$.

It isn't always clear just by looking at a quadratic equation whether the solutions will be real or not. For example, look at this quadratic equation:

$$x^2 - 12x + 41 = 0$$

We can always complete the square to find out what the solutions will be:

$$\begin{aligned} x^2 - 12x + 36 + 5 &= 0 \\ (x-6)^2 + 5 &= 0 \\ (x-6)^2 &= -5 \\ x - 6 &= \pm i\sqrt{5} \\ x &= 6 \pm i\sqrt{5} \end{aligned}$$

This equation has non-real, complex solutions $6 + i\sqrt{5}$ and $6 - i\sqrt{5}$.

Lesson 17 Practice Problems

1. Find the solution or solutions to each equation.

   a. $x^2 + 0.5x - 14 = 0$

   b. $x^2 + 12x + 36 = 0$

   c. $x^2 - 3x + 8 = 0$

   d. $x^2 + 4 = 0$

2. Which describes the solutions to the equation $x^2 + 7 = 0$?

   A. One real solution

   B. Two real solutions

   C. One non-real solution

   D. Two non-real solutions

3. Explain how you know $\sqrt{3x + 2} = -16$ has no solutions.

   (From Unit 3, Lesson 7.)

4. Determine the number of real solutions and non-real solutions to each equation. Use the graphs; don't do any calculations to find the solutions.

a. $x^2 - 6x + 7 = 0$

b. $3x^2 + 2x + 1 = 0$

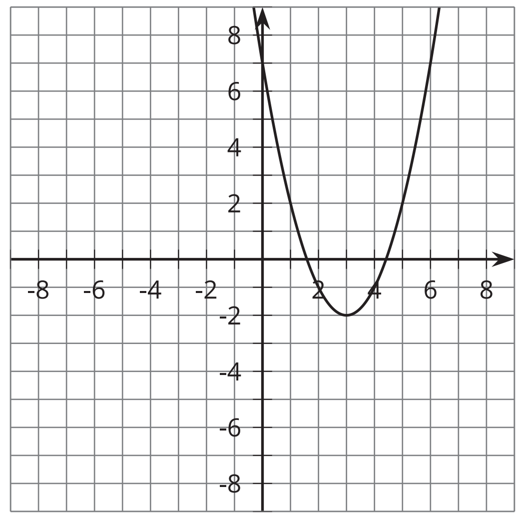
c. $-x^2 - 3x + 2 = 0$

d. $x^2 - 6x + 7 = -2$

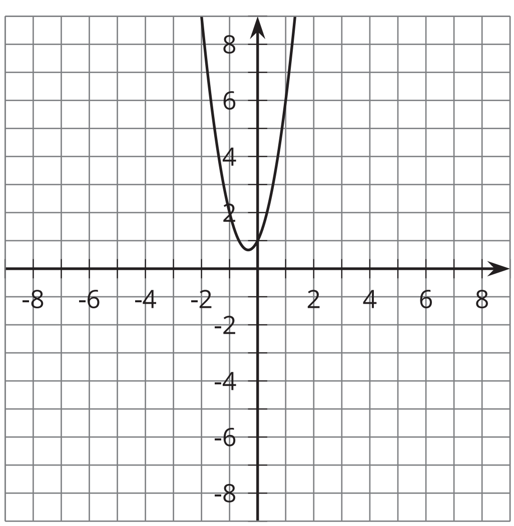
e. $-x^2 - 3x + 2 = 6$

f. $3x^2 + 2x + 1 = 2$

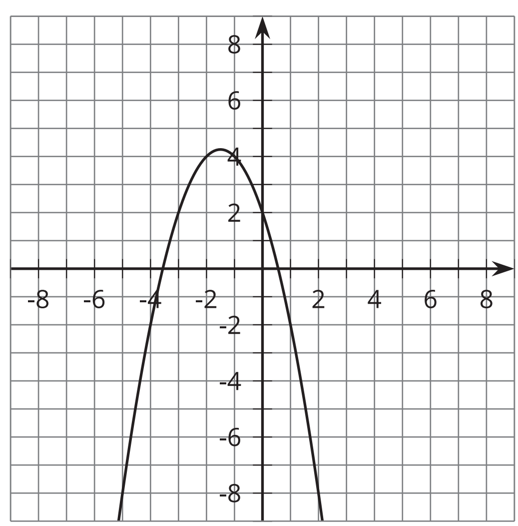
$y = x^2 - 6x + 7$



$y = 3x^2 + 2x + 1$



$y = -x^2 - 3x + 2$



5. a. Write $(5-5i)^2$ in the form $a+bi$, where a and b are real numbers.

b. Write $(5-5i)^4$ in the form $a+bi$, where a and b are real numbers.

(From Unit 3, Lesson 14.)

6. What number n makes this equation true?

$x^2+11x+\frac{121}{4}=(x+n)^2$

A. $\frac{11}{4}$

B. $\frac{11}{2}$

C. 11

D. $\frac{121}{4}$

(From Unit 3, Lesson 16.)

Lesson 18: The Quadratic Formula and Complex Solutions

- Let's use the quadratic formula to find complex solutions to quadratic equations.

18.1: Math Talk: Real or Not?

Mentally decide whether the solutions to each equation are real numbers or not.

$w^2 = -367$

$x^2 + 25 = 0$

$(y + 5)^2 = 0$

$(z + 5)^2 = -367$

18.2: Be Discriminating

Kiran was using the quadratic formula to solve the equation $x^2 - 12x + 41 = 0$. He wrote this:

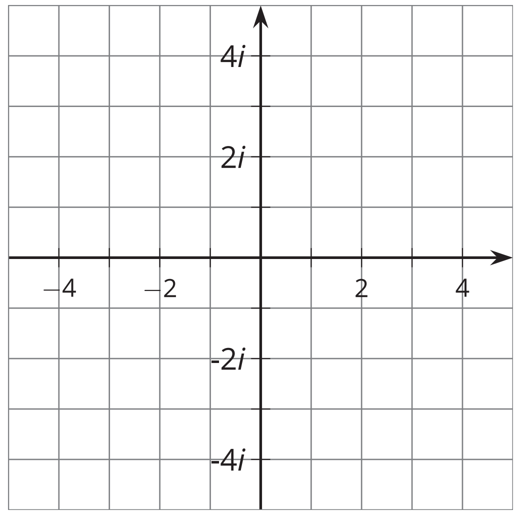
$$x = \frac{12 \pm \sqrt{144 - 164}}{2}$$

Then he noticed that the number inside the square root is negative and said, "This equation doesn't have any solutions."

1. Do you agree with Kiran? Explain your reasoning.

2. Write $\sqrt{-20}$ as an imaginary number.

3. Solve the equation $3x^2 - 10x + 50 = 0$ and plot the solutions in the complex plane.



Are you ready for more?

Although imaginary numbers let us describe complex solutions to quadratic equations, they were actually discovered and accepted because they could help us find real solutions to equations with polynomials of degree 3. In the 16th century, mathematicians discovered a cubic formula for solving equations of degree 3, but to use it they sometimes had to work with complex numbers. Let's see an example where this comes up.

1. To find a solution to the equation $x^3 - px - q = 0$ the cubic formula would first tell us to find a complex number, z, which is $\frac{q}{2} + i\sqrt{(\frac{p}{3})^3 - (\frac{q}{2})^2}$. Find z when our equation is $x^3 - 15x - 4 = 0$.

2. The next step is to find a complex number w so that $w^3 = z$. Show that $w = 2 + i$ works for the z we found in step 1.

3. If we write $w = a + bi$ where a and b are real numbers, the solutions to our equation are $2a$, -$a + b\sqrt{3}$, and -$a - b\sqrt{3}$. What are the three solutions to our equation $x^3 - 15x - 4 = 0$?

18.3: Solving All Kinds of Quadratics

For each row, you and your partner will each solve a quadratic equation. You should each get the same answer. If you disagree, work to reach agreement.

partner A	partner B
$x^2 - 4x - 4 = 0$	$(x-2)^2 = 8$
$(y-2)^2 = -8$	$y^2 - 4y + 12 = 0$
$(z + \frac{3}{2})^2 = -\frac{29}{4}$	$2z^2 + 6z = -19$
$w^2 + 3w = 5$	$(w + \frac{3}{2})^2 = \frac{29}{4}$
$4t^2 - 20t + 25 = 0$	$4(t^2 - 5t) = -25$

Lesson 18 Summary

Sometimes when we use the quadratic formula to solve a quadratic equation, we get a negative number inside the square root symbol. This means that the solutions to the equation must involve imaginary numbers. For example, consider the following equation:

$$5x^2 + x + 10 = 0$$

Using the quadratic formula, we know that:

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 5 \cdot 10}}{2 \cdot 5}$$

or

$$x = \frac{-1 \pm \sqrt{-199}}{10}$$

Which means that the two solutions are:

$$x = -\frac{1}{10} + \frac{\sqrt{199}}{10}i$$

and

$$x = -\frac{1}{10} - \frac{\sqrt{199}}{10}i$$

Lesson 18 Practice Problems

1. Clare solves the quadratic equation $4x^2 + 12x + 58 = 0$, but when she checks her answer, she realizes she made a mistake. Explain what Clare's mistake was.

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 4 \cdot 58}}{2 \cdot 4}$$
$$x = \frac{-12 \pm \sqrt{144 - 928}}{8}$$
$$x = \frac{-12 \pm \sqrt{-784}}{8}$$
$$x = \frac{-12 \pm 28i}{8}$$
$$x = -1.5 \pm 28i$$

2. Write in the form $a + bi$, where a and b are real numbers:

   a. $\frac{5 \pm \sqrt{-4}}{3}$

   b. $\frac{10 \pm \sqrt{-16}}{2}$

   c. $\frac{-3 \pm \sqrt{-144}}{6}$

3. Priya is using the quadratic formula to solve two different quadratic equations.

   For the first equation, she writes $x = \frac{4 \pm \sqrt{16-72}}{12}$

   For the second equation, she writes $x = \frac{8 \pm \sqrt{64-24}}{6}$

   Which equation(s) will have real solutions? Which equation(s) will have non-real solutions? Explain how you know.

4. Find the exact solution(s) to each of these equations, or explain why there is no solution.

   a. $x^2 = 25$

   b. $x^3 = 27$

   c. $x^2 = 12$

   d. $x^3 = 12$

(From Unit 3, Lesson 8.)

5. Kiran is solving the equation $\sqrt{x+2}-5=11$ and decides to start by squaring both sides. Which equation results if Kiran squares both sides as his first step?

   A. $x+2-25=121$

   B. $x+2+25=121$

   C. $x+2-10\sqrt{x+2}+25=121$

   D. $x+2+10\sqrt{x+2}+25=121$

   (From Unit 3, Lesson 9.)

6. Plot each number on the real or imaginary number line.

a. $-\sqrt{4}$

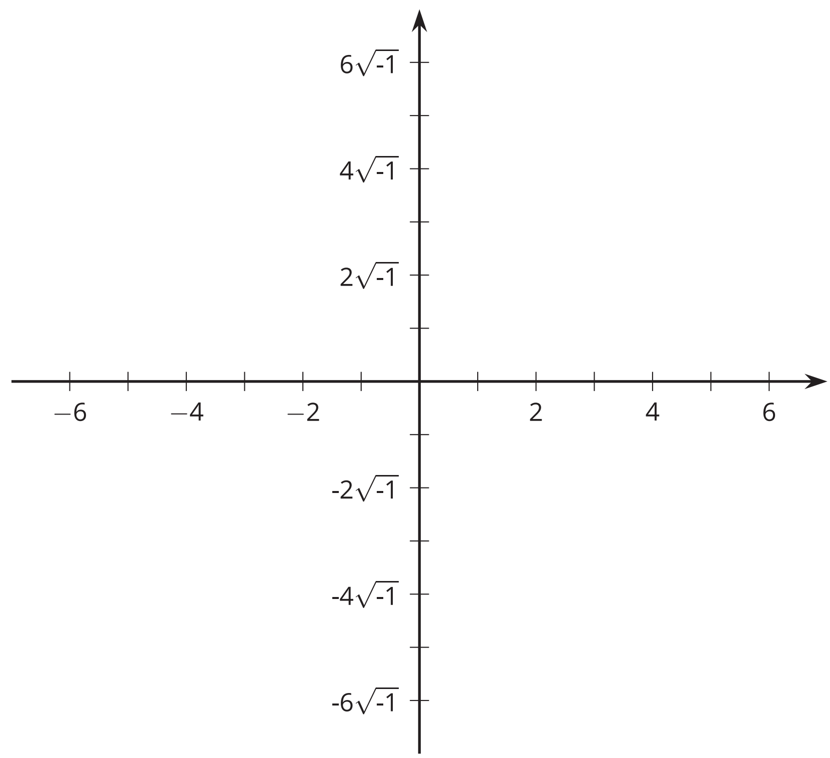
b. $\sqrt{-1}$

c. $3\sqrt{4}$

d. $-3\sqrt{-1}$

e. $4\sqrt{-1}$

f. $2\sqrt{2}$



(From Unit 3, Lesson 10.)