Lesson 16: Using Graphs and Logarithms to Solve Problems (Part 2)

- Let's compare exponential functions by studying their graphs.

16.1: Two Bank Accounts

A business owner opened two different types of investment accounts at the start of the year. The functions f and g represent the values of the two accounts as a function of the number of months after the accounts were opened.

1. Here are some true statements about the investment accounts. What does each statement mean?
    a. $f(3) > g(3)$

    b. $f(6) < g(6)$

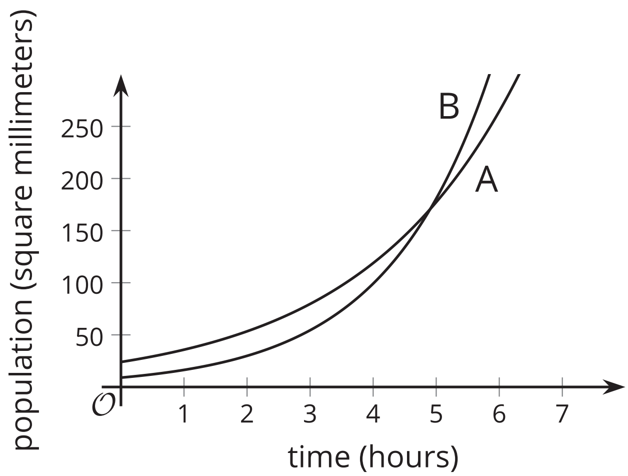
    c. $f(m) = g(m)$

2. If the two functions were graphed on the same coordinate plane, what might it look like? Sketch the two functions.

16.2: Bacteria in Different Conditions

To study the growth of bacteria in different conditions, a scientist measures the area, in square millimeters, occupied by two populations.

The growth of Population A, in square millimeters, can be modeled by $f(h) = 24 \cdot e^{(0.4h)}$ where h is the number of hours since the experiment began. The growth of Population B can be modeled by $g(h) = 9 \cdot e^{(0.6h)}$. Here are the graphs representing the two populations.



1. In this situation, what does the point of intersection of the two graphs tell us?

2. Suppose the population coordinate of the point of intersection is 171. Explain why we can find the corresponding time coordinate by:
   a. solving $f(h) = 171$ or $g(h) = 171$

   b. solving the equation $f(h) = g(h)$

3. Solve either $f(h) = 171$ or $g(h) = 171$. Show your reasoning.

4. Solve $f(h) = g(h)$. Show your reasoning.

Are you ready for more?

The functions f and g are given by $f(t) = 10e^{0.5t}$ and $g(t) = 8e^{0.4t}$.

1. Is there any positive value of t so that $f(t) = g(t)$? Explain how you know.

2. When do f and g reach the value 1000?

16.3: Populations of Two Countries

The population, in millions, of Country C is given by the equation $f(t) = 16 \cdot e^{(0.02t)}$. The population of Country D is given by $g(t) = 17.5 \cdot e^{(0.025t)}$. In both equations, t is the number of years since 1980.

1. Will there be a time when the two populations are equal? Explain or show your reasoning.

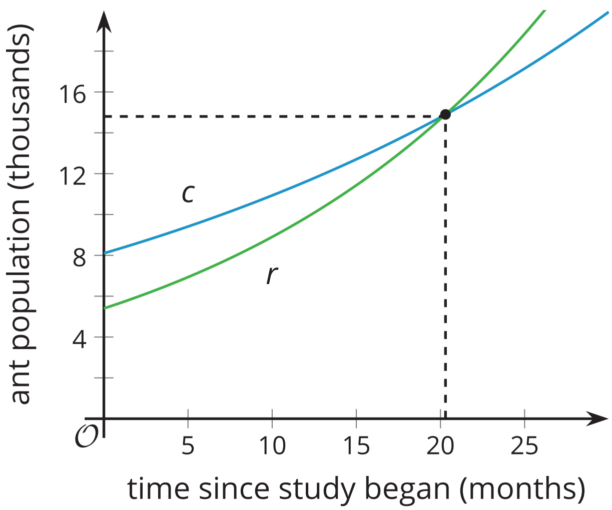
2. At some point in time, the population of Country C reached 30 million. When does this happen? Explain or show your reasoning.

Lesson 16 Summary

Graphs representing functions can help us visualize how two or more quantities are changing in a situation. Let's consider the populations of two colonies of ants.

The population, in thousands, of a colony of carpenter ants and a colony of red wood ants can be modeled with functions $c(x) = 8.1 \cdot e^{(0.03x)}$ and $r(x) = 5.4 \cdot e^{(0.05x)}$, respectively. Here, x is the time in months after the colonies were first studied.

From the equations, we can tell which colony had a greater initial population (carpenter ants, 8.1 thousand) and which had a greater growth factor (red wood ants, $e^{(0.05)}$). Will the colony of red wood ants eventually exceed that of the carpenter ants? If so, when might it happen? Graphs representing $y = c(x)$ and $y = r(x)$ can help us answer these questions.



The intersection of the graphs tells us that about 20 months after the study began, the two colonies have the same population, about 15 thousand. After that point, the population of red wood ants is greater than that of carpenter ants. To find out more exactly when the two colonies have the same population, we can use graphing technology to find the coordinates of the intersection point more precisely.

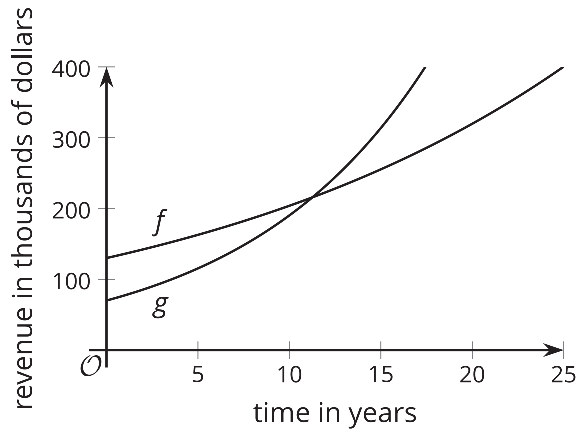
Another way to find the point of intersection is using the equations for the functions. At the point of intersection of the graphs, the two functions have the same y-value, so we can write the equation $8.1 \cdot e^{(0.03x)} = 5.4 \cdot e^{(0.05x)}$. Then we can solve this equation:

$$\begin{aligned} 8.1 \cdot e^{(0.03x)} &= 5.4 \cdot e^{(0.05x)} \\ \frac{8.1}{5.4} &= \frac{e^{(0.05x)}}{e^{(0.03x)}} \\ 1.5 &= e^{(0.02x)} \\ \ln(1.5) &= 0.02x \\ \frac{0.405}{0.02} &\approx x \\ 20.3 &\approx x \end{aligned}$$

This solution means that about 20.3 months after the study began, the two colonies have the same population.

Lesson 16 Practice Problems

1. The revenues of two companies can be modeled with exponential functions f and g. Here are the graphs of the two functions. In each function, the revenue is in thousands of dollars and time, t, is measured in years. The y-coordinate of the intersection is 215.7. Select **all** statements that correctly describe what the two graphs reveal about the revenues.



A. The intersection of the graphs tells us when the revenues of the two companies grow by the same factor.

B. The intersection tells us when the two companies have the same revenue.

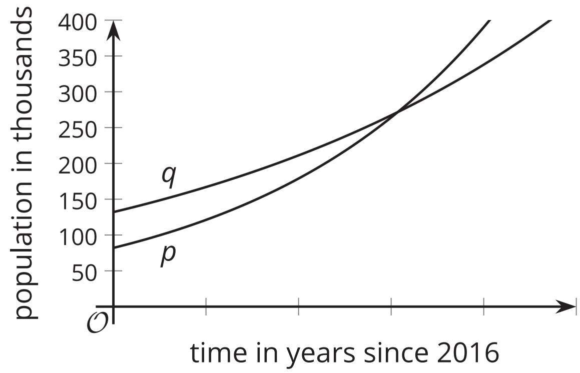
C. At the intersection, $f(t) > g(t)$.

D. At the intersection, $f(t) = 215.7$ and $g(t) = 215.7$.

E. We need to know both expressions that define f and g to find the value of t at the intersection.

F. If we know at least one of the expressions that define f and g, we can calculate the value of t at the intersection.

2. The population of a fast-growing city in Texas can be modeled with the equation $p(t) = 82 \cdot e^{(0.078t)}$. The population of a fast-growing city in Tennessee can be modeled with $q(t) = 132 \cdot e^{(0.047t)}$. In both equations, t represents years since 2016 and the population is measured in thousands. The graphs representing the two functions are shown. The point where the two graphs intersect has a y-coordinate of about 271.7.



a. What does the intersection mean in this situation?

b. Find the x-coordinate of the intersection point by solving each equation. Show your reasoning.

i. $p(t) = 271.7$

ii. $q(t) = 271.7$

c. Explain why we can find out the t value at the intersection of the two graphs by solving $p(t) = q(t)$.

3. The function f is given by $f(x) = 100 \cdot 3^x$. Select **all** equations whose graph meets the graph of f for a *positive* value of x.

   A. $y = 10 \cdot e^x$

   B. $y = 500 \cdot e^x$

   C. $y = 500 \cdot e^{-x}$

   D. $y = 1{,}000 \cdot 2^x$

   E. $y = 600 \cdot 10^x$

4. The half-life of nickel-63 is 100 years. A students says, “An artifact with nickel-63 in it will lose a quarter of that substance in 50 years.”

   Do you agree with this statement? Explain your reasoning.

   (From Unit 4, Lesson 7.)

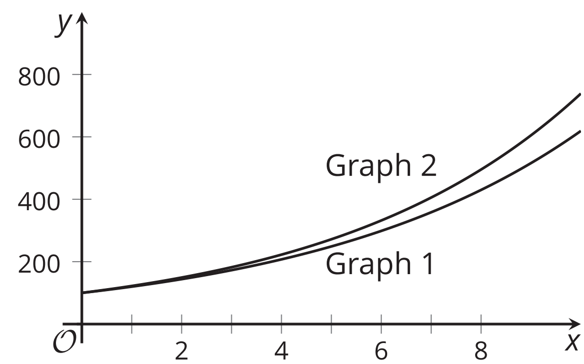
5. *Technology required.* Estimate the value of each expression and record it. Then, use a calculator to find its value and record it.

expression	estimate	calculator value
$\log 123$		
$\log 110{,}000$		
$\log 1.1$		

   (From Unit 4, Lesson 11.)

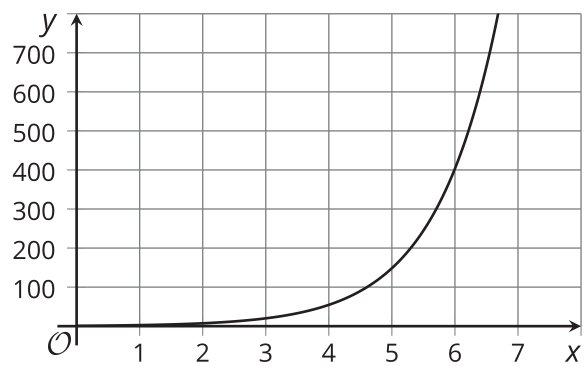
6. Here are graphs of the functions f and g given by $f(x) = 100 \cdot (1.2)^x$ and $g(x) = 100 \cdot e^{0.2x}$.

   Which graph corresponds to each function? Explain how you know.



(From Unit 4, Lesson 13.)

7. Here is a graph that represents $f(x) = e^x$.



   Explain how we can use the graph to estimate:

   a. The solution to an equation such as $300 = e^x$.

   b. The value of $\ln 700$.

(From Unit 4, Lesson 15.)

Lesson 17: Logarithmic Functions

- Let's graph log functions.

17.1: Which One Doesn't Belong: Functions

Which one doesn't belong? Be prepared to explain your reasoning.

$f(x) = 4 \cdot (0.75)^x$

$g(x) = 4 \cdot e^{(0.75x)}$

$h(x) = (0.75) \cdot 4^x$

$j(x) = 4 \cdot \log x$

17.2: How Long Will It Take?

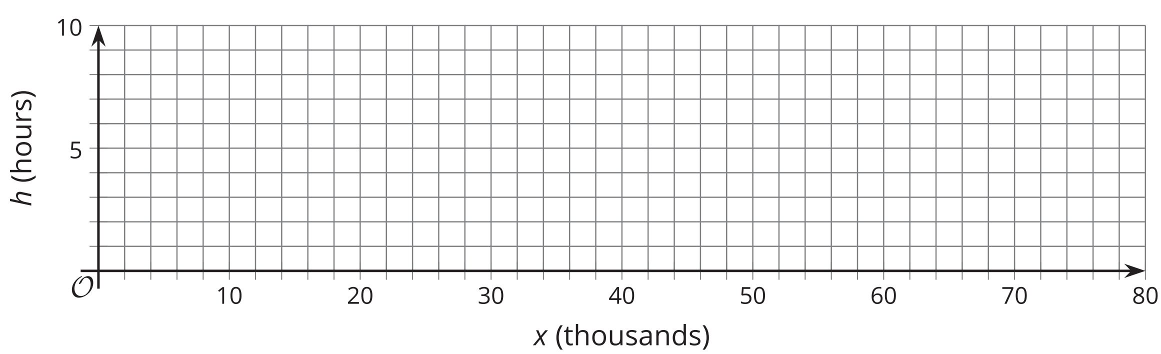
A colony of 1,000 bacteria doubles in population every hour.

1. Explain why we can write $h = \log_2 x$ to represent the number of hours, h, it takes for the one thousand bacteria to reach a population of x thousand.

2. Complete the table with the corresponding values of h.

x **(thousands)**	1	2	4	8	16	50	80
h **(hours)**							

3. Plot the pairs of values on the coordinate plane. Make two observations about the graph.



4. Use the graph to estimate the missing values in the table.

x (thousands)	10	24	72
h (hours)			

17.3: Another Logarithmic Function

Earlier we saw that $h = \log_2 x$ represents the number of hours for 1 thousand bacteria, doubling every hour, to reach a population of x, in thousands.

1. Suppose the function d, defined by $d(x) = \log_{10} x$, represents the number of days it takes 1 thousand of another species of bacteria to reach a population of x, in thousands. How is this population of bacteria growing?

2. Graph d using graphing technology. Make two observations about the graph.

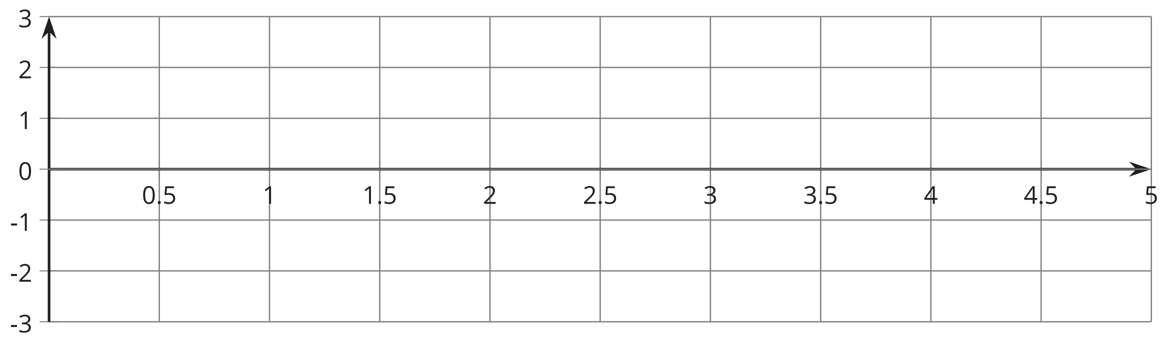
3. Use your graph to estimate the values of $d(50)$ and $d(20{,}000)$. (Adjust your graphing window as needed.) Explain what each value means in this situation.

4. Estimate or find the population after 5 days.

Are you ready for more?

1. Without graphing, how do you think the graphs of the equations $y = \log_2(x)$ and $y = \log_{10}(x)$ compare? Do they ever meet?

2. Graph both equations on the same axes to test your conjectures.

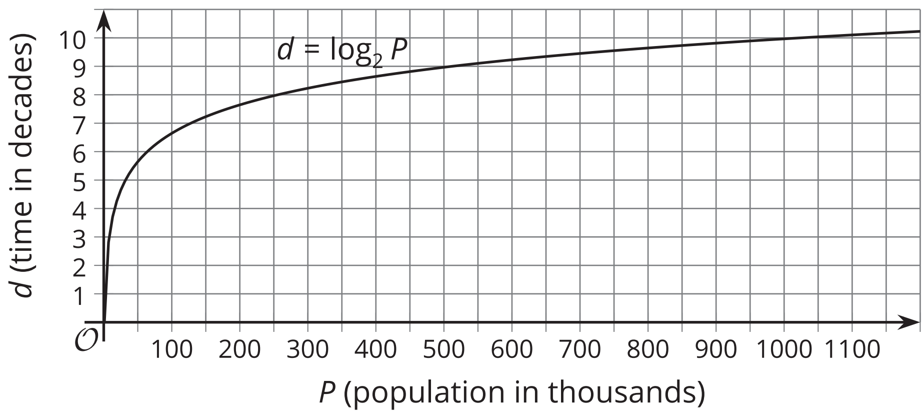


Lesson 17 Summary

Earlier we have studied exponential relationships where the input values are the exponent in the function. Sometimes we want to express an exponential relationship where the values we want to find, the outputs, are the exponents. A **logarithmic function** can help us do that.

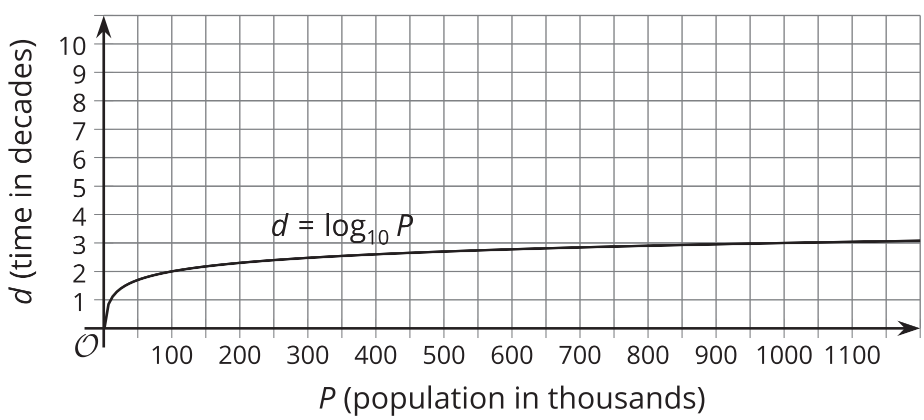
For example: Suppose the population of a town starts at one thousand and doubles every decade since first measured. We can write $P = 1 \cdot 2^d$ or $P = 2^d$ to represent the population, in thousands, after d decades.

But if we want to know how long, in decades, it would take to reach certain population sizes, in thousands, we can write a logarithmic function $d = \log_2 P$. In this function, the input is P, population in thousands, and the output is d, time in decades. Here is a graph representing that function.



We can use the graph to estimate the answer to a question such as, "How many decades will it take for the population to reach a million?" In this case, the answer is about 10 decades, because one million is 1,000 thousands and $\log_2 1{,}000 \approx 10$ (or, thinking in terms of powers of 2, we know that $2^{10} = 1{,}024$).

Suppose the population of that town expands by a factor of 10 every decade instead of by a factor of 2. The function representing the time it takes to reach a certain population, in thousands, would be $d = \log_{10} P$.



From the graph, we can see that it takes only 3 decades to reach 1,000 thousands, because $\log_{10} 1{,}000 = 3$ (or $10^3 = 1{,}000$).

Glossary

- logarithmic function

Lesson 17 Practice Problems

1. The relationship between a bacteria population p, in thousands, and time d, in days, since it was measured to be 1,000 can be represented by the equation $d = \log_2 p$.

   Select **all** statements that are true about the situation.

   A. Each day, the bacteria population grows by a factor of 2.

   B. The equation $p = 2^d$ also defines the relationship between the population in thousands and time in days.

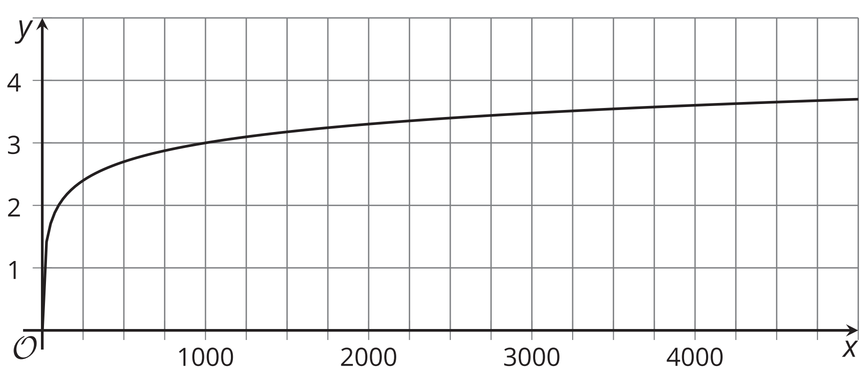
   C. The population reaches 7,000 after $\log_2 7{,}000$ days.

   D. The expression $\log_2 10$ tells us when the population reaches 10,000.

   E. The equation $d = \log_2 p$ represents a logarithmic function.

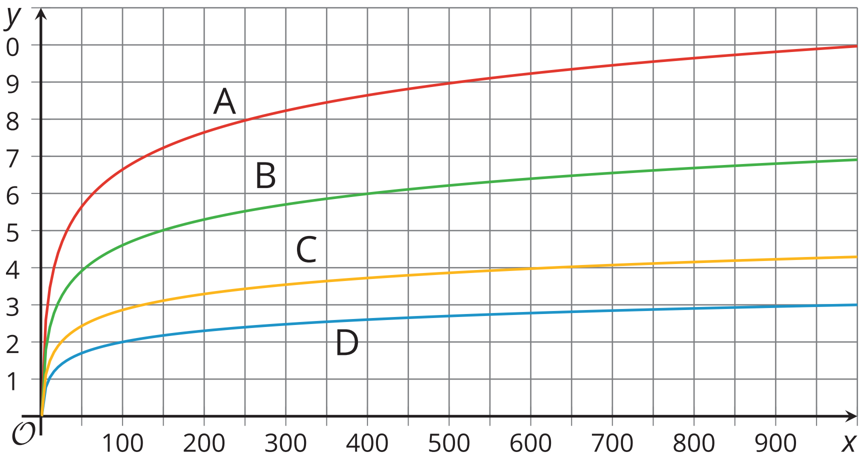
   F. The equation $7 = \log_2 128$ tells us that the population reaches 128,000 in 7 days.

2. Here is the graph of a logarithmic function.



   What is the base of the logarithm? Explain how you know.

3. Match each equation with a graph that represents it.



A. A

B. B

C. C

D. D

1. $f(x) = \log_2 x$

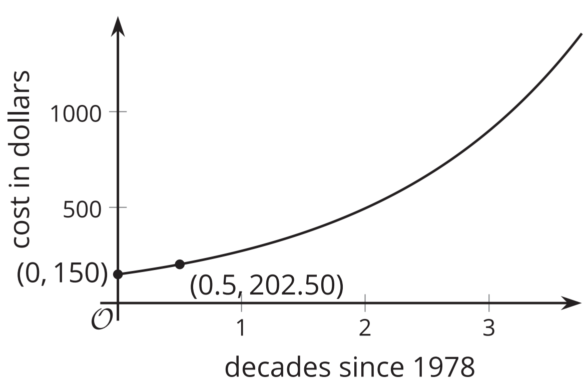
2. $g(x) = \log_{10} x$

3. $h(x) = \log_5 x$

4. $j(x) = \ln x$

4. The graph represents the cost of a medical treatment, in dollars, as a function of time, d, in decades since 1978.

The expression $150 \cdot (1.35)^3$ represents the cost of the medical treatment sometime after 1978. Which year does it represent?



A. 1986

B. 1993

C. 1998

D. 2018

(From Unit 4, Lesson 5.)

5. The equation $A(w) = 180 \cdot e^{(0.01w)}$ represents the area, in square centimeters, of a wall covered by mold as a function of w, time in weeks since the area was measured.

   Explain or show that we can approximate the area covered by mold in 8 weeks by multiplying $A(7)$ by 1.01.

   (From Unit 4, Lesson 13.)

6. Solve each equation without using a calculator. Some solutions will need to be expressed using log notation.

   a. $10^{(n-3)} = 10$

   b. $\frac{1}{2} \cdot 10^{x} = 0.05$

   c. $10^{\frac{1}{3}t} = 100$

   d. $10^{2x} = 48$

   (From Unit 4, Lesson 14.)

7. *Technology required.* The population of Mali can be represented by $m(t) = 17 \cdot e^{(0.03t)}$. The population of Saudi Arabia can be represented by $s(t) = 31 \cdot e^{(0.015t)}$. In both models, t represents years since 2014 and the populations are measured in millions.

   a. Which country had a higher population in 2014? Explain how you know.

   b. Which country has a higher growth rate? Explain how you know.

   c. Use graphing technology to graph both equations on the same axes.

   d. Do the two graphs intersect? If so, estimate their point of intersection and explain what it means in this situation. If not, explain what it means that the two graphs don't intersect.

(From Unit 4, Lesson 16.)