Lesson 14: Solving Exponential Equations

- Let's solve equations using logarithms.

14.1: A Valid Solution?

To solve the equation $5 \cdot e^{3a} = 90$, Lin wrote the following:

$$\begin{aligned} 5 \cdot e^{3a} &= 90 \\ e^{3a} &= 18 \\ 3a &= \log_e 18 \\ a &= \frac{\log_e 18}{3} \end{aligned}$$

Is her solution valid? Be prepared to explain what she did in each step to support your answer.

14.2: Natural Logarithm

1. Complete the table with equivalent equations. The first row is completed for you.

	exponential form	logarithmic form
a.	$e^0 = 1$	$\ln 1 = 0$
b.	$e^1 = e$	
c.	$e^{-1} = \frac{1}{e}$	
d.		$\ln \frac{1}{e^2} = -2$
e.	$e^x = 10$	

2. Solve each equation by expressing the solution using $\ln$ notation. Then, find the approximate value of the solution using the "ln" button on a calculator.
   a. $e^m = 20$

   b. $e^n = 30$

   c. $e^p = 7.5$

14.3: Solving Exponential Equations

Without using a calculator, solve each equation. It is expected that some solutions will be expressed using log notation. Be prepared to explain your reasoning.

1. $10^x = 10{,}000$

2. $5 \cdot 10^x = 500$

3. $10^{(x+3)} = 10{,}000$

4. $10^{2x} = 10{,}000$

5. $10^x = 315$

6. $2 \cdot 10^x = 800$

7. $10^{(1.2x)} = 4{,}000$

8. $7 \cdot 10^{(0.5x)} = 70$

9. $2 \cdot e^x = 16$

10. $10 \cdot e^{3x} = 250$

Are you ready for more?

1. Solve the equations $10^n = 16$ and $10^n = 2$. Express your answers as logarithms.

2. What is the relationship between these two solutions? Explain how you know.

Lesson 14 Summary

So far we have solved exponential equations by

- finding whole number powers of the base (for example, the solution of $10^x = 100{,}000$ is 5)
- estimation (for example, the solution of $10^x = 300$ is between 2 and 3)
- using a logarithm and approximating its value on a calculator (for example, the solution of $10^x = 300$ is $\log 300 \approx 2.48$)

Sometimes solving exponential equations takes additional reasoning. Here are a couple of examples.

$$\begin{aligned} 5 \cdot 10^x &= 45 \\ 5 \cdot 10^x &= 45 \\ 10^x &= 9 \\ x &= \log 9 \end{aligned}$$

$$\begin{aligned} 10^{(0.2t)} &= 1{,}000 \\ 10^{(0.2t)} &= 10^3 \\ 0.2t &= 3 \\ t &= \frac{3}{0.2} \\ t &= 15 \end{aligned}$$

In the first example, the power of 10 is multiplied by 5, so to find the value of x that makes this equation true each side was divided by 5. From there, the equation was rewritten as a logarithm, giving an exact value for x.

In the second example, the expressions on each side of the equation were rewritten as powers of 10: $10^{(0.2t)} = 10^3$. This means that the exponent $0.2t$ on one side and the 3 on the other side must be equal, and leads to a simpler expression to solve where we don't need to use a logarithm.

How do we solve an exponential equation with base e, such as $e^x = 5$? We can express the solution using the **natural logarithm**, the logarithm for base e. Natural logarithm is written as $\ln$, or sometimes as $\log_e$. Just like the equation $10^2 = 100$ can be rewritten, in logarithmic form, as $\log_{10} 100 = 2$, the equation $e^0 = 1$ and be rewritten as $\ln 1 = 0$. Similarly, $e^{-2} = \frac{1}{e^2}$ can be rewritten as $\ln \frac{1}{e^2} = -2$.

All this means that we can solve $e^x = 5$ by rewriting the equation as $x = \ln 5$. This says that x is the exponent to which base e is raised to equal 5.

To estimate the size of $\ln 5$, remember that e is about 2.7. Because 5 is greater than e^1, this means that $\ln 5$ is greater than 1. e^2 is about $(2.7)^2$ or 7.3. Because 5 is less than e^2, this means that $\ln 5$ is less than 2. This suggests that $\ln 5$ is between 1 and 2. Using a calculator we can check that $\ln 5 \approx 1.61$.

Glossary

- natural logarithm

Lesson 14 Practice Problems

1. Solve each equation without using a calculator. Some solutions will need to be expressed using log notation.

   a. $4 \cdot 10^x = 400{,}000$

   b. $10^{(n+1)} = 1$

   c. $10^{3n} = 1{,}000{,}000$

   d. $10^p = 725$

   e. $6 \cdot 10^t = 360$

2. Solve $\frac{1}{4} \cdot 10^{(d+2)} = 0.25$. Show your reasoning.

3. Write two equations—one in logarithmic form and one in exponential form—that represent the statement: “the natural logarithm of 10 is y”.

4. Explain why $\ln 1 = 0$.

5. If $\log_{10}(x) = 6$, what is the value of x? Explain how you know.

(From Unit 4, Lesson 9.)

6. For each logarithmic equation, write an equivalent equation in exponential form.

    a. $\log_2 16 = 4$

    b. $\log_3 9 = 2$

    c. $\log_5 5 = 1$

    d. $\log_{10} 20 = y$

    e. $\log_2 30 = y$

(From Unit 4, Lesson 10.)

7. The function f is given by $f(x) = e^{0.07x}$.

    a. What is the continuous growth rate of f?

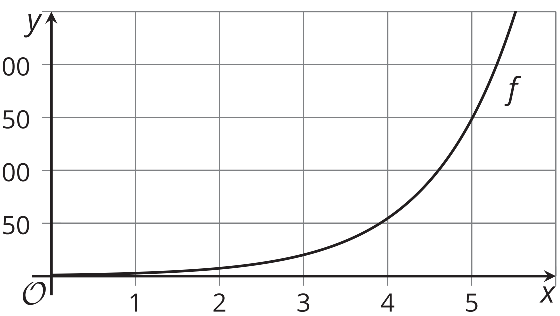
    b. By what factor does f grow when the input x increases by 1?

(From Unit 4, Lesson 13.)

Lesson 15: Using Graphs and Logarithms to Solve Problems (Part 1)

- Let’s use graphs and logarithms to solve problems.

15.1: Using a Graph to Estimate



Here is a graph that represents an exponential function with base e, defined by $f(x) = e^x$.

1. Explain how to use the graph to estimate logarithms such as $\ln 100$.

2. Use the graph to estimate $\ln 100$.

3. How can you use a calculator to check your estimate? What would you enter into the calculator?

15.2: Retire A Millionaire?

The expression $1 \cdot e^{(0.06t)}$ models the balance, in thousands of dollars, of an account t years after the account was opened.

1. What is the account balance:
    a. when the account is opened?

    b. after 1 year?

    c. after 2 years?

2. Diego says that the expression $\ln 5$ represents the time, in years, when the account will have 5 thousand dollars. Do you agree? Explain your reasoning.

3. Suppose you opened this account at the beginning of this year. Assume that you deposit no additional money and withdraw nothing from the account. Will the account balance reach $1,000,000 and make you a millionaire by the time you reach retirement? Show your reasoning.

Are you ready for more?

Noah is 15 years old and wants to retire a millionaire when he is 60. If he invests $1,000 today, what interest rate would he need to achieve this goal?

15.3: Cicada Population

A population of cicadas is modeled by a function defined by $f(w) = 250 \cdot e^{(0.5w)}$ where w is the number of weeks since the population was first measured.

1. Explain why solving the equation $500 = 250 \cdot e^{(0.5w)}$ gives the number of weeks it takes for the cicada population to double.

2. How many weeks does it take the cicada population to double? Show your reasoning.

3. Use graphing technology to graph $y = f(w)$ and $y = 100{,}000$ on the same axes. Explain why we can use the intersection of the two graphs to estimate when the cicada population will reach 100,000.

Lesson 15 Summary

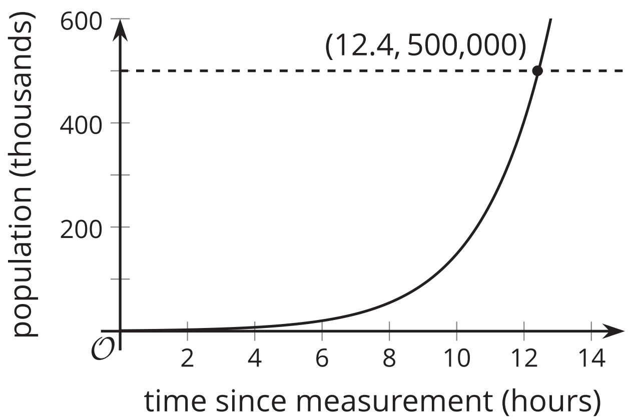
We can use the natural logarithm to solve exponential equations that are expressed with the base e.

Suppose a bacteria population is modeled by the equation $f(h) = 1{,}000 \cdot e^{(0.5h)}$, where h is the number of hours since the population was first measured. When will the population reach 500,000?

One way to answer this is to solve the equation $1{,}000 \cdot e^{(0.5h)} = 500{,}000$, which is when $e^{(0.5h)} = 500$.

The natural logarithm tells us the exponent to which we raise e to get a given number, so $0.5h = \ln 500$. This means $h = \frac{\ln 500}{0.5}$ or about 12.4, so it takes 12.4 hours (or 12 hours and 24 minutes) for the population to reach 500,000.

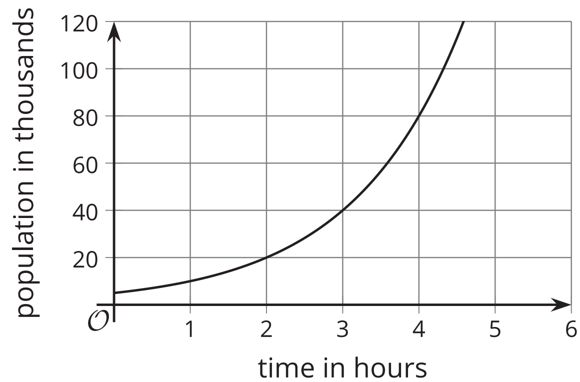
We can also use a graph to solve an exponential equation. To solve $1{,}000 \cdot e^{(0.5h)} = 500{,}000$, we can graph $y = 1{,}000 \cdot e^{(0.5h)}$ and $y = 500{,}000$ on the same coordinate plane and find the point of intersection.



The graph shows us that the bacteria population reaches 500,000 when the input value is a little over 12, or about 12 hours after the population was first measured.

Lesson 15 Practice Problems

1. The equation $p(h) = 5{,}000 \cdot 2^h$ represents a bacteria population as a function of time in hours. Here is a graph of the function p.



   a. Use the graph to determine when the population will reach 100,000.

   b. Explain why $\log_2 20$ also tells us when the population will reach 100,000.

2. *Technology required.* Population growth in the U.S. between 1800 and 1850, in millions, can be represented by the function f, defined by $f(t) = 5 \cdot e^{(0.028t)}$.

   a. What was the U.S. population in 1800?

   b. Use graphing technology to graph the equations $y = f(t)$ and $y = 20$. Adjust the graphing window to the following boundaries: $0 < x < 100$ and $0 < y < 40$.

   c. What is the point of intersection of the two graphs, and what does it mean in this situation?

3. The growth of a bacteria population is modeled by the equation $p(h) = 1{,}000e^{(0.4h)}$. For each question, explain or show how you know.

   a. How long does it take for the population to double?

   b. How long does it take for the population to reach 1,000,000?

4. What value of b makes each equation true?

   a. $\log_b 144 = 2$

   b. $\log_b 64 = 2$

   c. $\log_b 64 = 3$

   d. $\log_b 64 = 6$

   e. $\log_b \frac{1}{9} = -2$

   (From Unit 4, Lesson 10.)

5. Put the following expressions in order, from least to greatest.

   $\log_2 11 \quad \log_3 5 \quad \log_5 25 \quad \log_{10} 1{,}000 \log_2 5$

   (From Unit 4, Lesson 11.)

6. Solve $9 \cdot 10^{(0.2t)} = 900$. Show your reasoning.

(From Unit 4, Lesson 14.)

7. Explain why $\ln 4$ is greater than 1 but is less than 2.

(From Unit 4, Lesson 14.)