Lesson 12: The Number e

- Let's learn about the number e.

12.1: Matching Situations and Equations

Match each equation to a situation it represents. Be prepared to explain how you know. Not all equations have a match.

$f(t) = 400 \cdot (0.5)^{0.1t}$

$g(t) = 400 \cdot (1.25)^{0.1t}$

$h(t) = 400 \cdot (0.75)^{0.1t}$

$j(t) = 400 \cdot (2)^{10t}$

$k(t) = 400 \cdot (2)^{0.1t}$

1. A scientist begins an experiment with 400 bacteria in a petri dish. The population doubles every 10 hours. The function gives the number of bacteria t hours since the experiment began.

2. A patient takes 400 mg of a medicine. The amount of medicine in her bloodstream decreases by 25% every 10 hours. The function gives the amount of medicine left in her bloodstream after t hours of taking the medicine.

3. The half-life of a radioactive element is 10 years. There are 400 g of the element in a sample when it is first studied. The function gives the amount of the element remaining t years later.

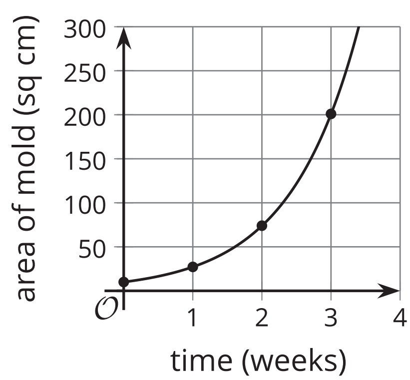
4. In a lake, the population of a species of fish is 400. The population is expected to grow by 25% in the next decade. The function gives the number of fish in the lake t years after it was 400.

12.2: Notice and Wonder: Moldy Growth

A spot of mold is found on a basement wall. Its area is about 10 square centimeters. Here are three representations of a function that models how the mold is growing.

time (weeks)	area of mold (sq cm)
0	10
1	27
2	74
3	201
4	546

$a(t) = 10 \cdot e^t$



What do you notice? What do you wonder?

12.3: $(1 + \text{tiny})^{\text{huge}}$

1. Here are some functions. For each function, describe, in words, the outputs for very tiny, positive values of x and for very large values of x.
   $a(x) = 1^x$

   $b(x) = -x$

   $d(x) = \frac{1}{x}$

   $f(x) = \left(\frac{1}{x}\right)^x$

   $g(x) = \left(1 + \frac{1}{x}\right)^x$

   $h(x) = e^x$

   $k(x) = 1 + x$

2. Remember that $e \approx 2.718$. What does the function g have to do with the number e?

3. What do you notice about the relationship between h and k for very small, positive values of x?

Are you ready for more?

Complete the table to show the value of each expression to the nearest hundred-thousandth. Two entries have already been completed as an example.

x	2^x	e^x	3^x
0.1	1.07177	1.10517	
0.01			
0.001			
0.0001			

What do you notice about the values in the table?

Lesson 12 Summary

Scientists, economists, engineers, and others often use the number e in their mathematical models. What is e?

e is an important constant in mathematics, just like the constant π, which is important in geometry. The value of e is approximately 2.718. Just like π, the number e is irrational, so it can't be represented as a fraction, and its decimal representation never repeats or terminates. The number is named after the 18th-century mathematician Leonhard Euler and is sometimes called *Euler's number*.

e has many useful properties and it arises in situations involving exponential growth or decay, so e often appears in exponential functions. In upcoming lessons, we will work with functions that are expressed using e.

Glossary

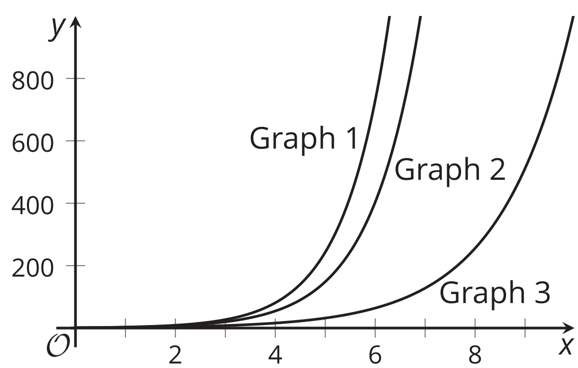
- e (mathematical constant)

Lesson 12 Practice Problems

1. Put the following expressions in order from least to greatest.

   - e^3
   - 2^2
   - e^2
   - $2e$
   - e^e

2. Here are graphs of three functions: $f(x) = 2^x$, $g(x) = e^x$, and $h(x) = 3^x$.



Which graph corresponds to each function? Explain how you know.

3. Which of the statements are true about the function f given by $f(x) = 100 \cdot e^{-x}$? Select **all** that apply.

   A. The y-intercept of the graph of f is at $(0, 100)$.

   B. The values of f increase when x increases.

   C. The value of f when $x = -1$ is a little less than 40.

   D. The value of f when $x = 5$ is less than 1.

   E. The value of f is never 0.

4. Suppose you have \$1 to put in an interest-bearing account for 1 year and are offered different options for interest rates and compounding frequencies (how often interest is calculated), as shown in the table. The highest interest rate is 100%, calculated once a year. The lower the interest rate, the more often it gets calculated.

   a. Complete the table with expressions that represent the amount you will have after one year, and then evaluate each expression to find its value in dollars (round to 5 decimal places).

interest rate	frequency per year	expression	value in dollars after 1 year
100%	1	$1 \cdot (1+1)^1$	
10%	10	$1 \cdot (1+0.1)^{10}$	
5%	20	$1 \cdot (1+0.05)^{20}$	
1%	100		
0.5%	200		
0.1%	1,000		
0.01%	10,000		
0.001%	100,000		

   b. Predict whether the account value will be greater than \$3 if there is an option for a 0.0001% interest rate calculated 1 million times a year. Check your prediction.

   c. What do you notice about the values of the account as the interest rate gets smaller and the frequency of compounding gets larger?

5. The function f is given by $f(x) = (1 + x)^{\frac{1}{x}}$. How do the values of f behave for small positive and large positive values of x?

6. Since 1992, the value of homes in a neighborhood has doubled every 16 years. The value of one home in the neighborhood was $136,500 in 1992.

   a. What is the value of this home, in dollars, in the year 2000? Explain your reasoning.

   b. Write an equation that represents the growth in housing value as a function of time in t years since 1992.

   c. Write an equation that represents the growth in housing value as a function of time in d decades since 1992.

   d. Use one of your equations to find the value of the home, in dollars, 1.5 decades after 1992.

(From Unit 4, Lesson 4.)

7. Write two equations—one in exponential form and one in logarithmic form—to represent each question. Use "?" for the unknown value.

   a. "To what exponent do we raise the number 5 to get 625?"

   b. "What is the log, base 3, of 27?"

(From Unit 4, Lesson 10.)

8. Clare says that $\log 0.1 = -1$. Kiran says that $\log(-10) = -1$. Do you agree with either one of them? Explain your reasoning.

(From Unit 4, Lesson 11.)

Lesson 13: Exponential Functions with Base e

- Let's look at situations that can be modeled using exponential functions with base e.

13.1: e on a Calculator

The other day, you learned that e is a mathematical constant whose value is approximately 2.718. When working on problems that involve e, we often rely on calculators to estimate values.

1. Find the e button on your calculator. Experiment with it to understand how it works. (For example, see how the value of $2e$ or e^2 can be calculated.)

2. Evaluate each expression. Make sure your calculator gives the indicated value. If it doesn't, check in with your partner to compare how you entered the expression.
    a. $10 \cdot e^{(1.1)}$ should give approximately 30.04166

    b. $5 \cdot e^{(1.1)(7)}$ should give approximately 11,041.73996

    c. $e^{\frac{9}{23}} + 7$ should give approximately 8.47891

13.2: Same Situation, Different Equations

The population of a colony of insects is 9 thousand when it was first being studied. Here are two functions that could be used to model the growth of the colony t months after the study began.

$$P(t) = 9 \cdot (1.02)^t \qquad\qquad Q(t) = 9 \cdot e^{(0.02t)}$$

1. Use technology to find the population of the colony at different times after the beginning of the study and complete the table.

t (time in months)	$P(t)$ (population in thousands)	$Q(t)$ (population in thousands)
6		
12		
24		
48		
100		

2. What do you notice about the populations in the two models?

3. Here are pairs of equations representing the populations, in thousands, of four other insect colonies in a research lab. The initial population of each colony is 10 thousand and they are growing exponentially. t is time, in months, since the study began.

Colony 1

$f(t) = 10 \cdot (1.05)^t$

$g(t) = 10 \cdot e^{(0.05t)}$

Colony 2

$k(t) = 10 \cdot (1.03)^t$

$l(t) = 10 \cdot e^{(0.03t)}$

Colony 3

$p(t) = 10 \cdot (1.01)^t$

$q(t) = 10 \cdot e^{(0.01t)}$

Colony 4

$v(t) = 10 \cdot (1.005)^t$

$w(t) = 10 \cdot e^{(0.005t)}$

   a. Graph each pair of functions on the same coordinate plane. Adjust the graphing window to the following boundaries to start: $0 < x < 50$ and $0 < y < 80$.

   b. What do you notice about the graph of the equation written using e and the counterpart written without e? Make a couple of observations.

13.3: e in Exponential Models

Exponential models that use e often use the format shown in this example:

output of the function

value of the function when t is 0

0.045 is the continuous growth rate. We can also say it's 4.5%, but the decimal number is used in the function.

$$P(t) = 13\,e^{0.045t}$$

the constant e (approximately 2.718)

input to the function (the independent variable)

Here are some situations in which a percent change is considered to be happening continuously. For each function, identify the missing information and the missing growth rate (expressed as a percentage).

1. At time $t = 0$, measured in hours, a scientist puts 50 bacteria into a gel on a dish. The bacteria are growing and the population is expected to show exponential growth.
   - function: $b(t) = 50 \cdot e^{(0.25t)}$
   - continuous growth rate per hour:

2. In 1964, the population of the United States was growing at a rate of 1.4% annually. That year, the population was approximately 192 million. The model predicts the population, in millions, t years after 1964.
   - function: $p(t) = \_\_\_\_\_\_\_\_ \cdot e^{\_\_\_\_t}$
   - continuous growth rate per year: 1.4%

3. In 1955, the world population was about 2.5 billion and growing. The model predicts the population, in billions, t years after 1955.
   - function: $q(t) = \_\_\_\_\_\_\_\_ \cdot e^{(0.0168t)}$
   - continuous growth rate per year:

13.4: Graphing Exponential Functions with Base e

1. Use graphing technology to graph the function defined by $f(t) = 50 \cdot e^{(0.25t)}$. Adjust the graphing window as needed to answer these questions:
   a. The function f models the population of bacteria in t hours after it was initially measured. About how many bacteria were in the dish 10 hours after the scientist put the initial 50 bacteria in the dish?

   b. About how many hours did it take for the number of bacteria in the dish to double? Explain or show your reasoning.

2. Use graphing technology to graph the function defined by $p(t) = 192 \cdot e^{(0.014t)}$. Adjust the graphing window as needed to answer these questions:
   a. The equation models the population, in millions, in the U.S. t years after 1964. What does the model predict for the population of the U.S. in 1974?

   b. In which year does the model predict the population will reach 300 million?

Are you ready for more?

Research what the population of the U.S. was in the year the model predicted 300 million people. How far off was the model? What factors do you think account for the actual population in that year being different from the prediction of the model? In what year did the U.S. actually reach 300 million people?

Lesson 13 Summary

Suppose 24 square feet of a pond is covered with algae and the area is growing at a rate of 8% each day.

We learned earlier that the area, in square feet, can be modeled with a function such as $a(d) = 24 \cdot (1 + 0.08)^d$ or $a(d) = 24 \cdot (1.08)^d$, where d is the number of days since the area was 24 square feet. This model assumes that the growth rate of 0.08 happens once each day.

In this lesson, we looked at a different type of exponential function, using the base e. For the algae growth, this might look like $A(d) = 24 \cdot e^{(0.08d)}$. This model is different because the 8% growth is not just applied at the end of each day: it is successively divided up and applied at every moment. Because the growth is applied at every moment or "continuously," the functions a and A are not the same, but the smaller the growth rate the closer they are to each other.

Many functions that express real-life exponential growth or decay are expressed in the form that uses e. For the algae model A, 0.08 is called the *continuous growth rate* while $e^{0.08}$ is the growth factor for 1 day. In general, when we express an exponential function in the form $P \cdot e^{rt}$, we are assuming the growth rate (or decay rate) r is being applied continuously and e^r is the growth (or decay) factor. When r is small, e^{rt} is close to $(1 + r)^t$.

Lesson 13 Practice Problems

1. The population of a town is growing exponentially and can be modeled by the equation $f(t) = 42 \cdot e^{(0.015t)}$. The population is measured in thousands, and time is measured in years since 1950.

   a. What was the population of the town in 1950?

   b. What is the approximate percent increase in the population each year?

   c. According to this model, approximately what was the population in 1960?

2. The revenue of a technology company, in thousands of dollars, can be modeled with an exponential function whose starting value is $395,000 where time t is measured in years after 2010.

   Which function predicts exactly 1.2% of annual growth: $R(t) = 395 \cdot e^{(0.012t)}$ or $S(t) = 395 \cdot (1.012)^t$? Explain your reasoning.

3. How are the functions f and g given by $f(x) = (1.05)^x$ and $g(x) = e^{0.05x}$ similar? How are they different?

4. a. A bond is worth \$100 and grows in value by 4 percent each year. Explain why the value of the bond after t years is given by $100 \cdot 1.04^t$.

   b. A second bond is worth \$100 and grows in value by 2 percent each half year. Explain why the value of the bond after t years is given by $100 \cdot (1.02)^{2t}$.

   c. A third bond is worth \$100 and grows in value by 4 percent each year, but the interest is applied continuously, at every moment. The value of this bond after t years is given by $100 \cdot e^{(0.04t)}$. Order the bonds from slowest growing to fastest growing. Explain how you know.

5. The population of a country is growing exponentially, doubling every 50 years. What is the annual growth rate? Explain or show your reasoning.

(From Unit 4, Lesson 6.)

6. Which expression has a greater value: $\log_3 \frac{1}{3}$ or $\log_b \frac{1}{b}$? Explain how you know.

(From Unit 4, Lesson 11.)

7. The expression $5 \cdot \left(\frac{1}{2}\right)^d$ models the amount of a radioactive substance, in nanograms, in a sample over time in decades, d. (1 nanogram is a billionth or 1×10^{-9} gram.)

   a. What do the 5 and the $\frac{1}{2}$ tell us in this situation?

   b. When will the sample have less than 0.5 nanogram of the radioactive substance? Express your answer to the nearest half decade. Show your reasoning.

   c. Show that only about 5 picograms of the substance will remain one century after the sample is measured. (A picogram is a trillionth or 1×10^{-12} gram.)

(From Unit 4, Lesson 7.)

8. Select **all** true statements about the number e.

A. e is a rational number.

B. e is approximately 2.718.

C. e is an irrational number.

D. e is between π and $\sqrt{2}$ on the number line.

E. e is exactly 2.718.

(From Unit 4, Lesson 12.)