Lesson 11: Introducing the Number i

- Let's meet i.

11.1: Math Talk: Squared

Find the value of each expression mentally.

$\left(2\sqrt{3}\right)^2$

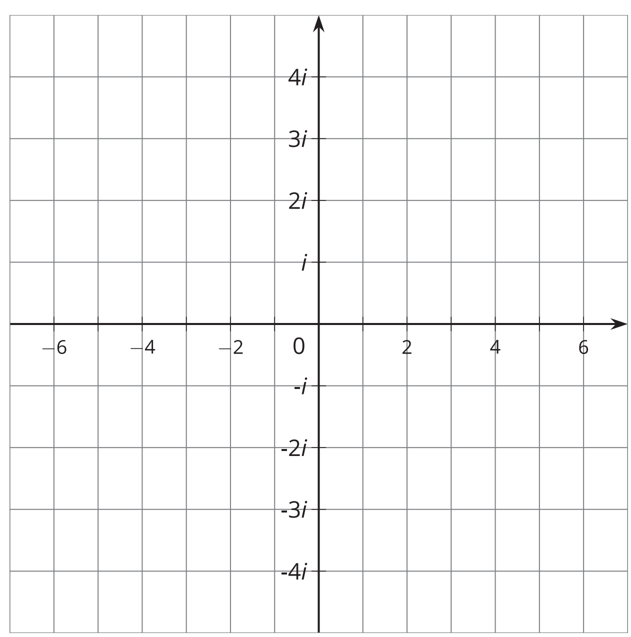
$\left(\frac{1}{2}\sqrt{3}\right)^2$

$\left(2\sqrt{-1}\right)^2$

$\left(\frac{1}{2}\sqrt{-1}\right)^2$

11.2: It is i

Find the solutions to these equations, then plot the solutions to each equation on the imaginary or real number line.



1. $a^2 = 16$

2. $b^2 = -9$

3. $c^2 = -5$

11.3: The i's Have It

Write these imaginary numbers using the number i.

1. $\sqrt{-36}$

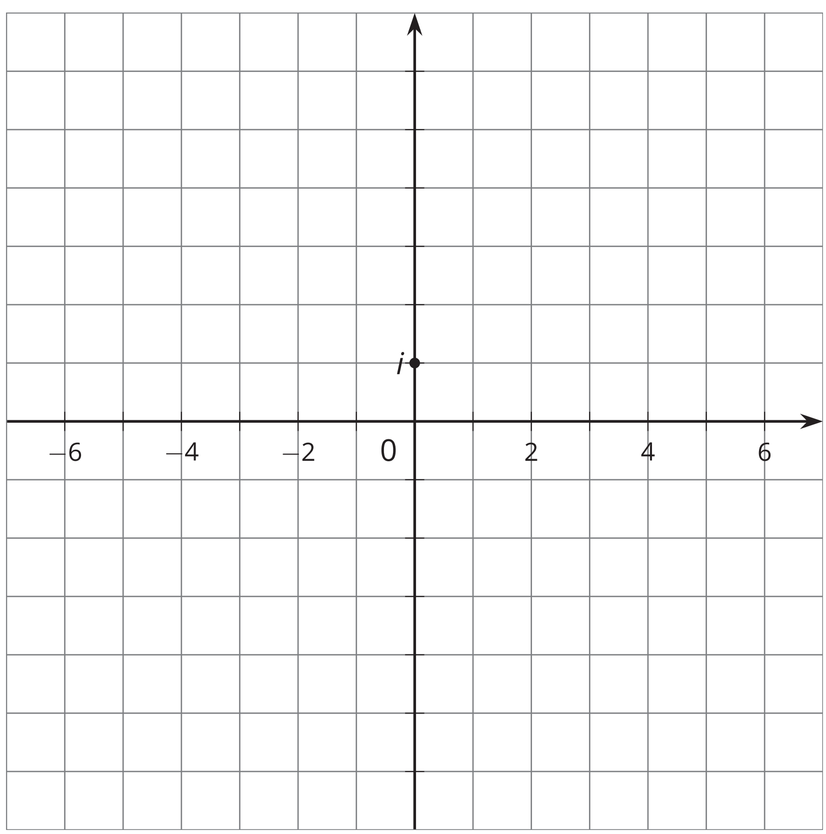
2. $\sqrt{-10}$

3. $-\sqrt{-100}$

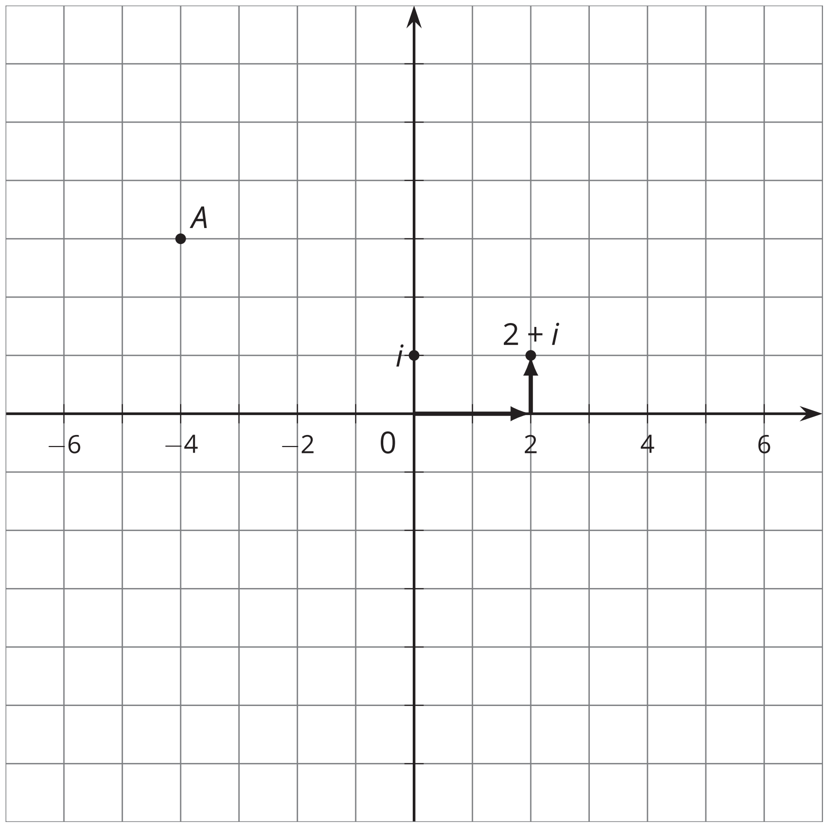
4. $-\sqrt{-17}$

11.4: Complex Numbers

1. Label at least 8 different imaginary numbers on the imaginary number line.



2. When we add a real number and an imaginary number, we get a **complex number**. The diagram shows where $2 + i$ is in the complex number plane. What complex number is represented by point A?



3. Plot these complex numbers in the complex number plane and label them.

   a. $-2 - i$

   b. $-6 + 3i$

   c. $5 + 4i$

   d. $1 - 3i$

Are you ready for more?

Diego says that all real numbers and all imaginary numbers are complex numbers but not all complex numbers are imaginary or real. Do you agree with Diego? Explain your reasoning.

Lesson 11 Summary

A square root of a number a is a number whose square is a. In other words, it is a solution to the equation $x^2 = a$. Every positive real number has two *real* square roots. For example, look at the number 35. Its square roots are $\sqrt{35}$ and $-\sqrt{35}$, because those are the two numbers that square to make 35 (remember, the $\sqrt{}$ symbol is defined to indicate the *positive* square root). In other words, $\left(\sqrt{35}\right)^2 = 35$ and $\left(-\sqrt{35}\right)^2 = 35$.

Similarly, every negative real number has two *imaginary* square roots. The two square roots of -1 are written i and $-i$. That means that

$$i^2 = -1$$

and

$$(-i)^2 = -1$$

Another example would be the number -17. Its square roots are $i\sqrt{17}$ and $-i\sqrt{17}$, because

$$\begin{aligned}\left(i\sqrt{17}\right)^2 &= 17i^2 \\ &= -17\end{aligned}$$

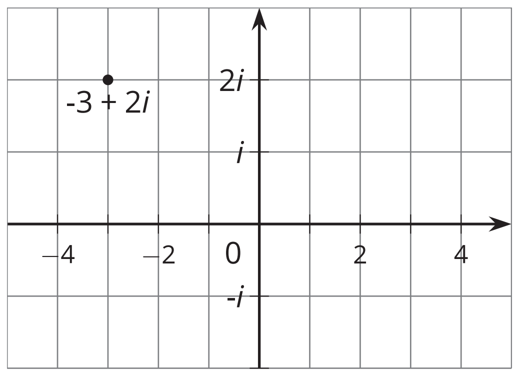
and

$$\begin{aligned}\left(-i\sqrt{17}\right)^2 &= 17(-i)^2 \\ &= 17i^2 \\ &= -17\end{aligned}$$

In general, if a is a positive real number, then the square roots of $-a$ are $i\sqrt{a}$ and $-i\sqrt{a}$.

Rarely, we might see something like $\sqrt{-17}$. It's not immediately clear which of the two square roots it is supposed to represent. By convention, $\sqrt{-17}$ is defined to indicate the square root on the positive imaginary axis, so $\sqrt{-17} = i\sqrt{17}$.

When we add a real number and an imaginary number, we get a **complex number**. Together, the real number line and the imaginary number line form a coordinate system that can be used to represent any complex number as a point in the *complex plane*. For example, the point shown represents the complex number $-3 + 2i$.



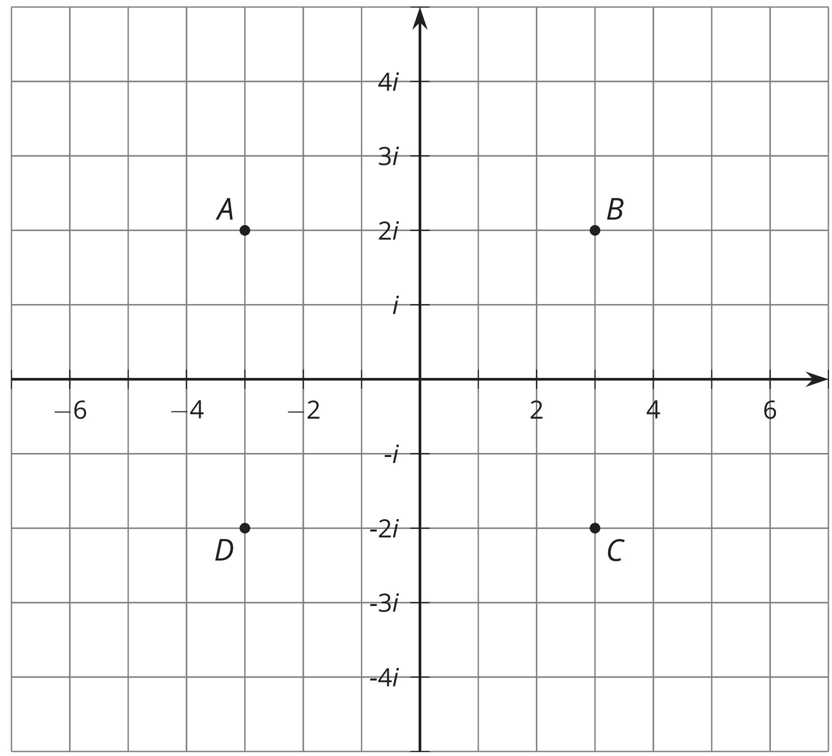
In this context, people call the real number line the *real axis* and the imaginary number line the *imaginary axis*. This is different than the coordinate plane you have seen before because those points were pairs of real numbers, like $(-3, 2)$, but in the complex plane, each point represents a single complex number. Note that since the real number line is part of the complex plane, real numbers are a special type of complex number. For example, the real number 5 can be described as the point $5 + 0i$ in the complex plane.

Glossary

- complex number

Lesson 11 Practice Problems

1. Which point represents the complex number $-3 + 2i$?



A. A

B. B

C. C

D. D

2. Match each expression to an equivalent expression.

A. $2i \cdot 8$	1. -16
B. $16i^3$	2. 16
C. $(2i)^4$	3. $-16i$
D. $2i \cdot 8i$	4. $16i$

3. a. Diego squared a number and got 4. Andre squared a different number and got 4. What were the numbers that Diego and Andre squared?

b. Jada squared a number and got -4. Elena squared a different number and got -4. What were the numbers that Jada and Elena squared?

4. Find **all** solutions to each equation.

a. $a^2 = 1$

b. $b^2 = 13$

c. $c^2 = -9$

d. $d^2 = -5$

5. Find the exact solution(s) to each of these equations, or explain why there is no solution.

   a. $\sqrt[3]{a+2} = 4$

   b. $\sqrt[3]{b} + 5 = 4$

   c. $\sqrt[3]{c-1} - 14 = -4$

(From Unit 3, Lesson 8.)

6. Explain how you know that $\sqrt{-1}$ is not a negative number.

(From Unit 3, Lesson 10.)

Lesson 12: Arithmetic with Complex Numbers

- Let’s work with complex numbers.

12.1: Math Talk: Telescoping Sums

Find the value of these expressions mentally.

$2 - 2 + 20 - 20 + 200 - 200$

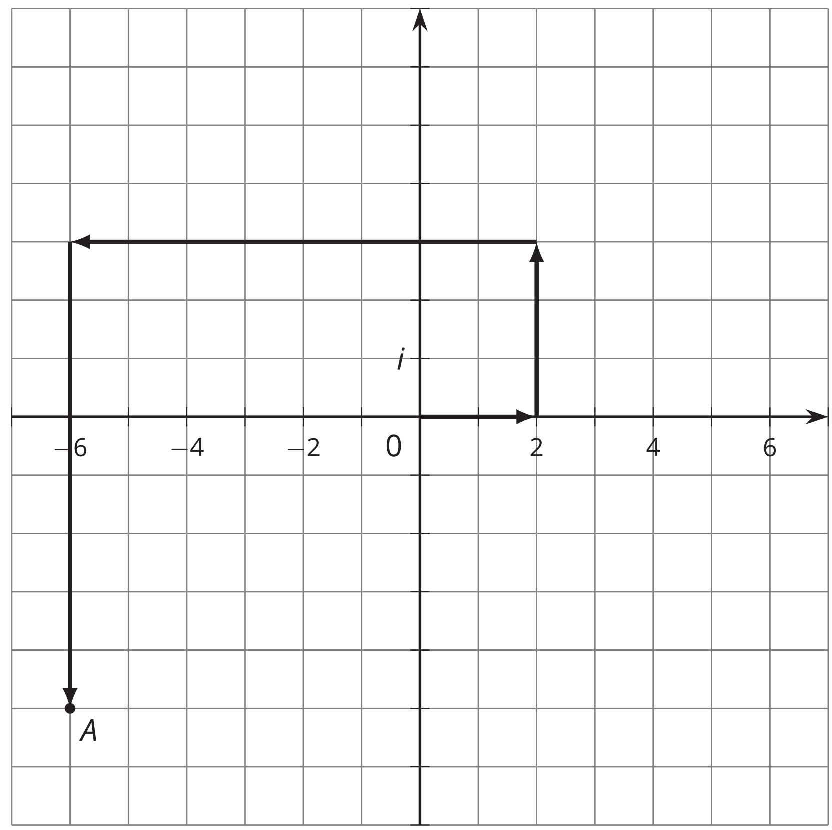
$100 - 50 + 10 - 10 + 50 - 100$

$3 + 2 + 1 + 0 - 1 - 2 - 3$

$1 + 2 + 4 + 8 + 16 + 32 - 16 - 8 - 4 - 2 - 1$

12.2: Adding Complex Numbers

1. This diagram represents $(2 + 3i) + (-8 - 8i)$.



a. How do you see $2 + 3i$ represented?

b. How do you see $-8 - 8i$ represented?

c. What complex number does A represent?

d. Add “like terms” in the expression $(2 + 3i) + (-8 - 8i)$. What do you get?

2. Write these sums and differences in the form $a + bi$, where a and b are real numbers.

a. $(-3 + 2i) + (4 - 5i)$ (Check your work by drawing a diagram.)

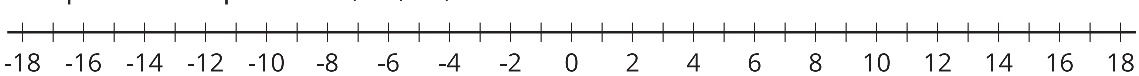
b. $(-37 - 45i) + (11 + 81i)$

c. $(-3 + 2i) - (4 - 5i)$

d. $(-37 - 45i) - (11 + 81i)$

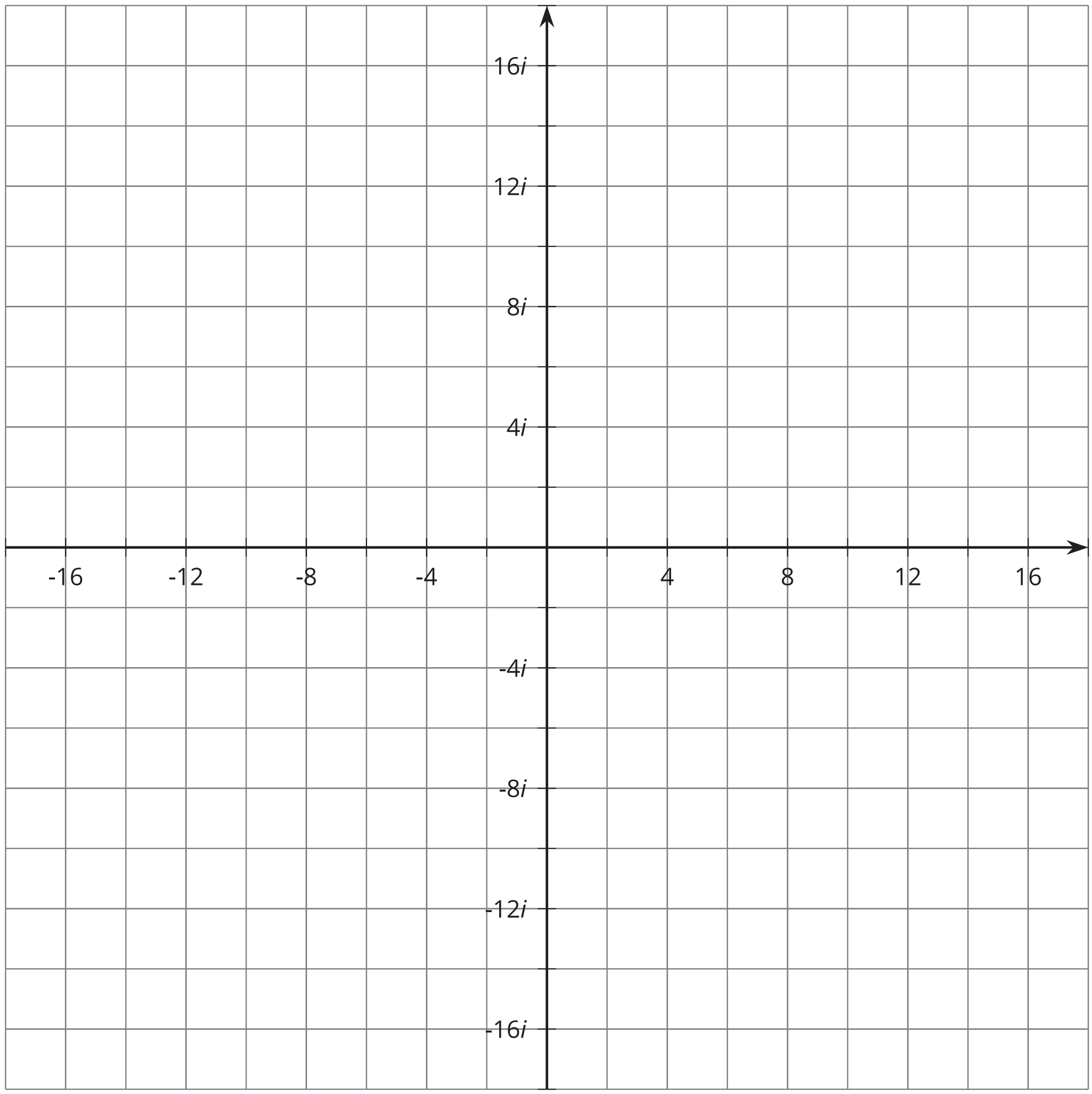
12.3: Multiplication on the Complex Plane

1. Draw points to represent 2, 2^2, 2^3, and 2^4 on the real number line.



2. a. Write $2i$, $(2i)^2$, $(2i)^3$, and $(2i)^4$ in the form $a+bi$.

b. Plot $2i$, $(2i)^2$, $(2i)^3$, and $(2i)^4$ on the complex plane.



Are you ready for more?

1. If a and b are positive numbers, is it true that $\sqrt{ab} = \sqrt{a}\sqrt{b}$? Explain how you know.

2. If a and b are negative numbers, is it true that $\sqrt{ab} = \sqrt{a}\sqrt{b}$? Explain how you know.

Lesson 12 Summary

When we add a real number with an imaginary number, we get a complex number. We usually write complex numbers as:

$$a + bi$$

where a and b are real numbers. We say that a is the real part and bi is the imaginary part.

To add (or subtract) two complex numbers, we add (or subtract) the real parts and add (or subtract) the imaginary parts. For example:

$$(2 + 3i) + (4 + 5i) = (2 + 4) + (3i + 5i) = 6 + 8i$$

$$(2 + 3i) - (4 + 5i) = (2 - 4) + (3i - 5i) = \text{-}2 - 2i$$

In general:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

When we raise an imaginary number to a power, we can use the fact that $i^2 = \text{-}1$ to write the result in the form $a + bi$. For example, $(4i)^3 = 4i \cdot 4i \cdot 4i$. We can group the i factors together to see how to rewrite this.

$$\begin{aligned} 4i \cdot 4i \cdot 4i &= (4 \cdot 4 \cdot 4) \cdot (i \cdot i \cdot i) \\ &= 64 \cdot (i^2 \cdot i) \\ &= 64 \cdot \text{-}1 \cdot i \\ &= \text{-}64i \end{aligned}$$

So in this example, a is 0 and b is -64.

Lesson 12 Practice Problems

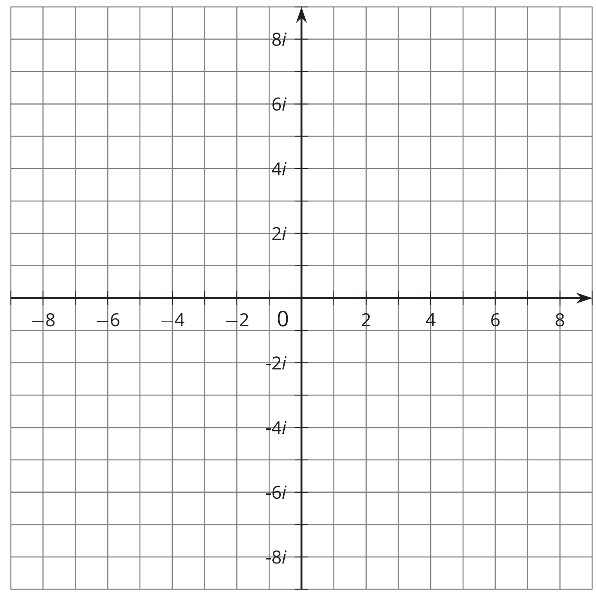
1. Write each expression in the form $a + bi$, where a and b are real numbers. You may plot the numbers in the complex plane as a guide.

   a. $2 \cdot \sqrt{-4}$

   b. $3i \cdot 2i$

   c. i^4

   d. $4 - 3\sqrt{-1}$



2. Which expression is equivalent to $(3 + 9i) - (5 - 3i)$?

   A. $-2 - 12i$

   B. $-2 + 12i$

   C. $15 + 27i$

   D. $15 - 27i$

3. What are a and b when you write $\sqrt{-16}$ in the form $a + bi$, where a and b are real numbers?

   A. $a = 0$, $b = -4$

   B. $a = 0$, $b = 4$

   C. $a = -4$, $b = 0$

   D. $a = 4$, $b = 0$

4. Fill in the boxes to make a true statement:

$$(\boxed{} - 3i) - (15 + \boxed{}\,i) = 7 - 12i$$

5. Plot each number on the real number line, or explain why the number is not on the real number line.

   a. $\sqrt{16}$

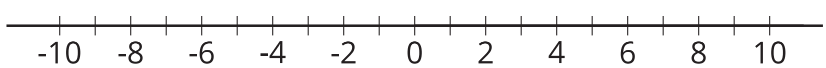
   b. $-\sqrt{16}$

   c. $\sqrt{-16}$

   d. $56^{1/2}$

   e. $-56^{1/2}$

   f. $(-56)^{1/2}$



(From Unit 3, Lesson 10.)

6. Which expression is equivalent to $\sqrt{-4}$?

   A. $-2i$

   B. $-4i$

   C. $2i$

   D. $4i$

(From Unit 3, Lesson 11.)