Lesson 5: Changes Over Rational Intervals

- Let's look at how an exponential function changes when the input changes by a fractional amount.

5.1: Changes Over Intervals

Consider the exponential function $h(x) = 4^x$. For each question, be prepared to share your reasoning with the class.

1. By what factor does h increase when the exponent x increases by 1?
2. By what factor does h increase when the exponent x increases by 2?
3. By what factor does h increase when the exponent x increases by 0.5?

5.2: Machine Depreciation

After purchase, the value of a machine depreciates exponentially. The table shows its value as a function of years since purchase. If a spreadsheet tool is available, consider using it to help you reason about the following questions.

years since purchase	value in dollars
0	16,000
0.5	
1	13,600
1.5	
2	11,560
3	9,826

1. The value of the machine in dollars is a function f of time t, the number of years since the machine was purchased. Find an equation defining f and be prepared to explain your reasoning.

2. Find the value of the machine when t is 0.5 and 1.5. Record the values in the table.

3. Observe the values in the table. By what factor did the value of the machine change:
   a. every one year, say from 1 year to 2 years, or from 0.5 years to 1.5 years?

   b. every half a year, say from 0 to 0.5 year, or from 1.5 years to 2 years?

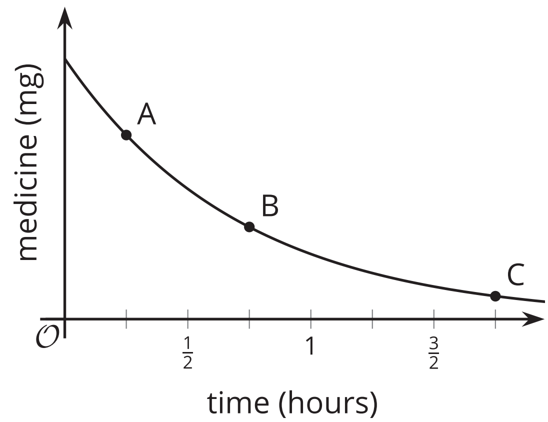
4. Suppose we know $f(q)$, the value of the machine q years since purchase. Explain how we could use $f(q)$ to find $f(q+0.5)$, the value of the machine half a year after that point.

Are you ready for more?

A bank account is growing exponentially. At the beginning of 2010, the balance, in dollars, was 1,200. At the beginning of 2015, the balance was 1,350. What would the bank account balance be at the beginning of 2018? Give an approximate answer as well as an exact expression.

5.3: Fever Medicine

The graph shows the amount of medicine in a child's body h hours after taking the medicine. The amount of medicine decays exponentially.



1. After $\frac{1}{4}$ hour there are about 7 mg of medicine left. After $\frac{3}{4}$ hour there are about 3.5 mg of medicine left. About how many mg of medicine are left after $1\frac{3}{4}$ hours? Explain how you know.

2. How does the decay rate from $\frac{1}{4}$ hour to $\frac{1}{2}$ hour compare to the decay rate from $\frac{1}{2}$ hour to $\frac{3}{4}$ hour? Explain how you know.

Lesson 5 Summary

Earlier we learned that, for an exponential function, every time the input increases by a certain amount the output changes by a certain factor.

For example, the population of a country, in millions, can be modeled by the exponential function $f(c) = 5 \cdot 16^c$, where c is time in centuries since 1900. By this model, the growth factor for any one century after the initial measurement is 16.

What about the growth factor for any one decade (one tenth of a century)? Let's start by finding the growth factors between 1910 and 1920 (c between 0.1 and 0.2) and between 1960 and 1970 (c between 0.6 and 0.7). To do that we can calculate the quotients of the function at those input values.

- from 1910 to 1920: $\dfrac{f(0.2)}{f(0.1)} = \dfrac{5 \cdot 16^{(0.2)}}{5 \cdot 16^{(0.1)}}$, which equals $16^{(0.1)}$ (or $\sqrt[10]{16}$)
- from 1960 to 1970: $\dfrac{f(0.7)}{f(0.6)} = \dfrac{5 \cdot 16^{(0.7)}}{5 \cdot 16^{(0.6)}}$, which equals $16^{(0.1)}$ (or $\sqrt[10]{16}$)

Now we can generalize about the growth factor for *any* one decade using the population x centuries after 1900, $f(x)$, and the population one decade (one tenth of a century) after that point, $f(x+0.1)$.

- from x to $(x+0.1)$: $\dfrac{f(x+0.1)}{f(x)} = \dfrac{5 \cdot 16^{(x+0.1)}}{5 \cdot 16^x}$, which also equals $16^{(0.1)}$ (or $\sqrt[10]{16}$)

This is consistent with what we know about how exponential functions change over whole-number intervals: they always increase or decrease by equal factors over equal intervals. This is true even when the intervals are fractional.

Lesson 5 Practice Problems

1. The table shows the monthly revenue of a business rising exponentially since it opened an online store.

months since online store opened	monthly revenue in dollars
0	72,000
1	
3	90,000
4	
6	112,500

a. Describe how the monthly revenue is growing.

b. Write an equation to represent the revenue, R, as a function of months, m, since the online store opened.

c. Find the monthly revenue 1 month after the online store opened. Record the value in the table. Explain your reasoning.

d. Explain how we can use the value of $R(1)$ to find $R(4)$.

2. At 7 a.m., a colony of 100 bacteria is placed on a petri dish where the population will triple every 6 hours.

   Select **all** statements that are true about the bacteria population.

   A. When the bacteria population reaches 900, 12 hours have passed since the colony was placed on the petri dish.

   B. Three hours after the colony is placed on the petri dish, there are 200 bacteria.

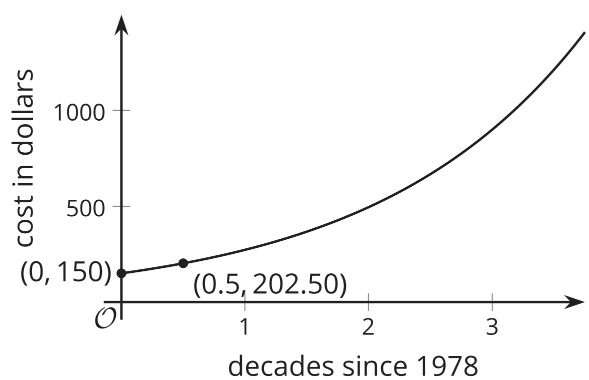
   C. Three hours after the colony is placed on the petri dish, there are about 173 bacteria in the colony.

   D. In the first hour the colony is placed on the petri dish, the population grows by a factor of $3^{\frac{1}{6}}$.

   E. Between 8 a.m. and 9 a.m., the population grows by a factor of $3^{\frac{2}{3}}$.

3. The graph represents the cost of a medical treatment, in dollars, as a function of time, d, in decades since 1978.

   Find the cost of the treatment, in dollars, when $d = 1$. Show your reasoning.



4. The exponential function f is given by $f(x) = 3^x$.

    a. By what factor does f increase when the exponent x increases by 1? Explain how you know.

    b. By what factor does f increase when the exponent x increases by 2? Explain how you know.

    c. By what factor does f increase when the exponent x increases by $\frac{1}{2}$? Explain how you know.

5. A piece of paper has area 93.5 square inches. How many times does it need to be folded in half before the area is less than 1 square inch? Explain how you know.

    (From Unit 4, Lesson 1.)

6. The area covered by an invasive tropical plant triples every year. By what factor does the area covered by the plant increase every month?

    (From Unit 4, Lesson 4.)

Lesson 6: Writing Equations for Exponential Functions

- Let's decide what information we need to write an equation for an exponential function.

6.1: All Equivalent?

1. Discuss with a partner why the following expressions are equivalent.
   a. $64^{\frac{1}{3}}$
   b. $\left(8^2\right)^{\frac{1}{3}}$
   c. $\left(2^3\right)^{\frac{2}{3}}$
   d. $\left(8^{\frac{1}{3}}\right)^2$
   e. $8^{\frac{2}{3}}$
2. What is another expression equivalent to these?

6.2: Info Gap: Two Points

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the data card:

1. Silently read the information on your card.
2. Ask your partner "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card. (Do not figure out anything for your partner!)
3. Before telling your partner the information, ask "Why do you need to know (that piece of information)?"
4. Read the problem card, and solve the problem independently.
5. Share the data card, and discuss your reasoning.

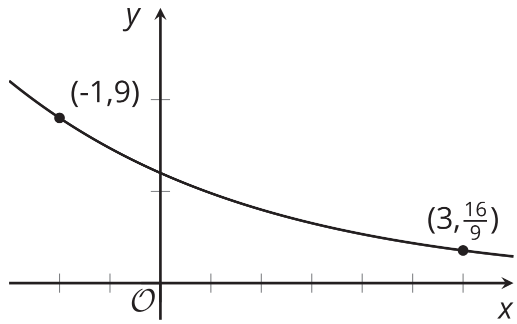
If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card, and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

Here is a graph representing a function of the form $h(x) = ab^x$.



Find the values of a and b. Show your reasoning.

6.3: Bacteria Growth Expressions

A bacteria population starts at 1000 and doubles every 10 hours.

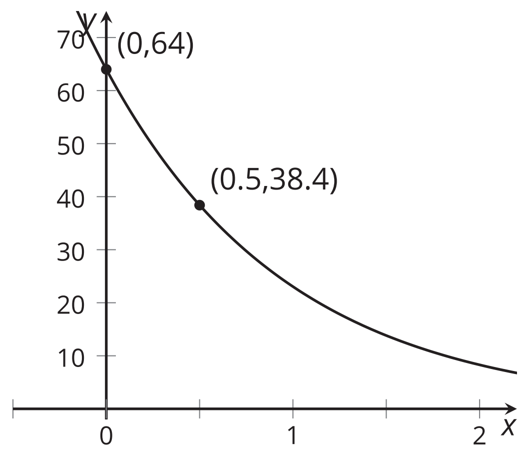
1. Explain why the expressions $1000 \cdot \left(2^{\frac{1}{10}}\right)^h$ and $1000 \cdot 2^{\frac{h}{10}}$ both represent the bacteria population after h hours.

2. By what factor does the bacteria population grow each hour? Explain how you know.

Lesson 6 Summary

Equations are helpful for communicating how quantities are changing. We can write equations from descriptions or from graphs.

Sometimes, the information on how a quantity is changing is given in a graph instead of in words. We can find an equation for an exponential function using two points on its graph, just as we've done in the past with linear functions. Let's say we want to find a function f of the form $f(x) = a \cdot b^x$ whose graph contains $(0, 64)$ and $(0.5, 38.4)$.



Since the point $(0, 64)$ is on the graph of f, we know the value of the function at 0 is 64. This means that $f(0) = a \cdot b^0 = a$, so the value of a is 64.

Using the second given point, $(0.5, 38.4)$, we know $f(0.5) = 38.4$. This means that $64 \cdot b^{0.5} = 38.4$. Solving this equation we have:

$$\begin{aligned} 64 \cdot b^{0.5} &= 38.4 \\ b^{0.5} &= \frac{38.4}{64} \\ b^{0.5} &= 0.6 \end{aligned}$$

To determine the exact value of b, let's use the properties of exponents. Since b is positive, we can show that $b = 0.36$ because

$$\begin{aligned} b^{0.5} &= 0.6 \\ \left(b^{0.5}\right)^2 &= (0.6)^2 \\ b &= 0.36 \end{aligned}$$

We can now write an equation defining f: $f(x) = 64 \cdot (0.36)^x$.

Lesson 6 Practice Problems

1. A population of 1,500 insects grows exponentially by a factor of 3 every week. Select **all** equations that represent or approximate the population, p, as a function of time in days, t, since the population was 1,500.

   A. $p(t) = 1{,}500 \cdot 3^t$

   B. $p(t) = 1{,}500 \cdot 3^{\frac{t}{7}}$

   C. $p(t) = 1{,}500 \cdot 3^7 t$

   D. $p(t) = 1{,}500 \cdot \left(3^{\frac{1}{7}}\right)^t$

2. The tuition at a public university was $21,000 in 2008. Between 2008 and 2010, the tuition had increased by 15%. Since then, it has continued to grow exponentially.

   Select **all** statements that describe the growth in tuition cost.

   A. The tuition cost can be defined by the function $f(y) = 21{,}000 \cdot (1.15)^{\frac{y}{2}}$, where y represents years since 2008.

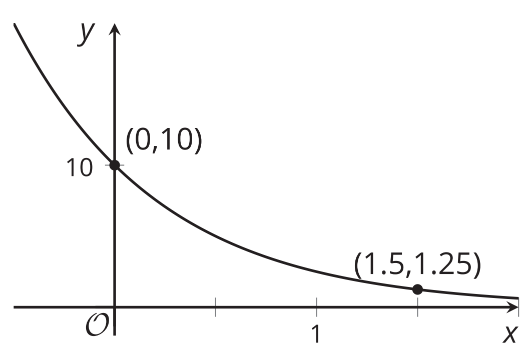
   B. The tuition cost increased 7.5% each year.

   C. The tuition cost increased about 7.2% each year.

   D. The tuition cost roughly doubles in 10 years.

   E. The tuition cost can be approximated by the function $f(d) = 21{,}000 \cdot 2^d$, where d represents decades since 2008.

3. Here is a graph that represents $g(x) = a \cdot b^x$. Find the values of a and b. Show your reasoning.



4. The number of fish in a lake is growing exponentially. The table shows the values, in thousands, after different numbers of years since the population was first measured.

years	population
0	10
1	
2	40
3	
4	
5	
6	

a. By what factor does the population grow every two years? Use this information to fill out the table for 4 years and 6 years.

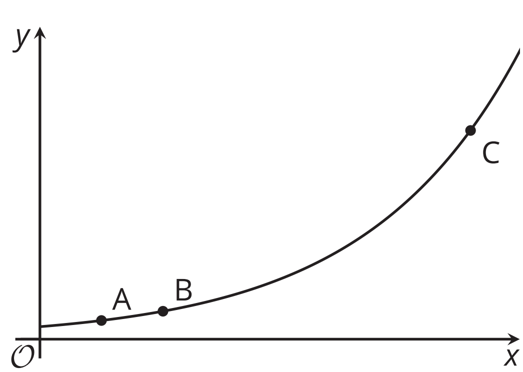
b. By what factor does the population grow every year? Explain how you know, and use this information to complete the table.

(From Unit 4, Lesson 3.)

5. The value of a home increases by 7% each year. Explain why the value of the home doubles approximately once each decade.

(From Unit 4, Lesson 4.)

6. Here is the graph of an exponential function f.



The coordinates of A are $\left(\frac{1}{4}, 3\right)$. The coordinates of B are $\left(\frac{1}{2}, 4.5\right)$. If the x-coordinate of C is $\frac{7}{4}$, what is its y-coordinate? Explain how you know.

(From Unit 4, Lesson 5.)