Lesson 3: Understanding Rational Inputs

- Let's look at exponential functions where the input values are not whole numbers.

3.1: Keeping Equations True

1. Select **all** solutions to $x \cdot x = 5$. Be prepared to explain your reasoning.
    a. $\frac{1}{25}$
    b. $\sqrt{5}$
    c. $\frac{5}{2}$
    d. $5^{\frac{1}{2}}$
    e. $\frac{\sqrt{5}}{2}$
    f. $\sqrt{25}$

2. Select **all** solutions to $p \cdot p \cdot p = 10$. Be prepared to explain your reasoning.
    a. $10^{\frac{1}{3}}$
    b. $\sqrt{10}$
    c. $\frac{10}{3}$
    d. $\frac{\sqrt{10}}{3}$
    e. $\sqrt[3]{10}$
    f. $\frac{1}{3}\sqrt{10}$

3.2: Florida in the 1800's

In 1840, the population of Florida was about 54,500. Between 1840 and 1860, the population grew exponentially, increasing by about 60% each decade.

1. Find the population of Florida in 1850 and in 1860 according to this model.

2. The population is a function f of the number of decades d after 1840. Write an equation for f.

3. a. Explain what $f(0.5)$ means in this situation.

   b. Graph your function using graphing technology and estimate the value of $f(0.5)$.

   c. Explain why we can find the value of $p(0.5)$ by multiplying 54,500 by $\sqrt{1.6}$. Find that value.

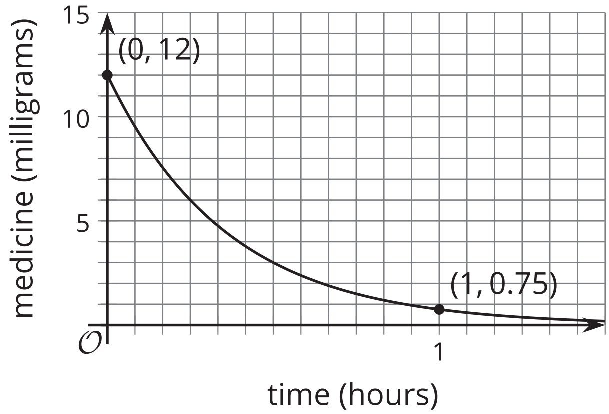
4. Based on the model, what was the population of Florida in 1858? Show your reasoning.

Are you ready for more?

Andre said, "The population of Florida increased by the same percentage between 1842 and 1852 and between 1847 and 1857." Do you agree with his statement? Explain or show your reasoning.

3.3: Disappearing Medicine

The amount of a medicine in the bloodstream of a patient decreases roughly exponentially. Here is a graph representing f, an exponential function that models the medicine in the body of a patient, t hours after an injection is given.



1. Use the graph to estimate $f\left(\frac{1}{3}\right)$ and explain what it tells us in this situation.

2. After one hour, 0.75 mg of medicine remains in the bloodstream. Find an equation that defines f.

Are you ready for more?

By what percentage does the amount of medicine in the body decay every 10 minutes? Explain or show your reasoning.

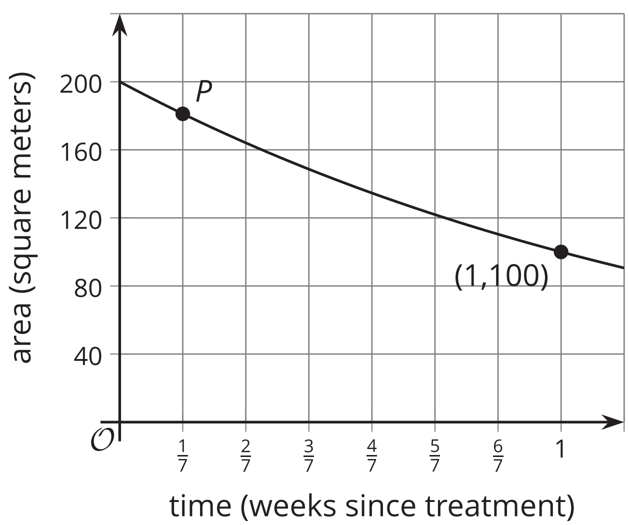
Lesson 3 Summary

Some exponential functions can have inputs that are any numbers on the number line, not just integers.

Suppose the area of a pond covered by algae A, in square meters, is modeled by $A = 200 \cdot \left(\frac{1}{2}\right)^w$ where w is the number of weeks since a treatment was applied to the pond. How could we use this equation to determine the area covered after 1 day?

Well, since $w = 1$ is one week and each week has 7 days, $w = \frac{1}{7}$ is 1 day. So after 1 day, the algae covers $200\left(\frac{1}{2}\right)^{\frac{1}{7}}$ square meters, or about 181 square meters. Using a calculator, we know that the expression $\left(\frac{1}{2}\right)^{\frac{1}{7}}$, which is equivalent to $\sqrt[7]{\frac{1}{2}}$, is about about 0.906. This means that after 1 day, only 91% of the algae from the previous day remains.

This information can also be seen on a graph representing the area. The point at $(1, 100)$ marks the area covered by the algae after 1 week. Point P marks the covered area after $\frac{1}{7}$ of a week or one day.



The graph can be used to estimate the vertical coordinate of P and shows that it is close to 180.

Lesson 3 Practice Problems

1. Select **all** solutions to $m \cdot m \cdot m = 729$.

   A. $\sqrt{729}$

   B. $\frac{729}{3}$

   C. $\frac{\sqrt{729}}{3}$

   D. $\frac{1}{3}\sqrt{729}$

   E. $729^{\frac{1}{3}}$

   F. $\sqrt[3]{729}$

2. In a pond, the area that is covered by algae doubles each week. When the algae was first spotted, the area it covered was about 12.5 square meters.

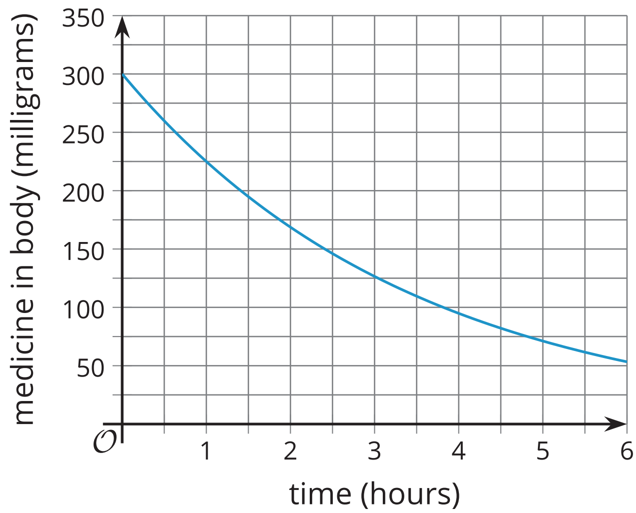
   a. Find the area, in square meters, covered by algae 10 days after it was spotted. Show your reasoning.

   b. Explain why we can find the area covered by algae 1 day after it was spotted by multiplying 12.5 by $\sqrt[7]{2}$.

3. The function m, defined by $m(h) = 300 \cdot \left(\frac{3}{4}\right)^h$, represents the amount of a medicine, in milligrams, in a patient's body. h represents the number of hours after the medicine is administered.

   a. What does $m(0.5)$ represent in this situation?

   b. This graph represents the function m. Use the graph to estimate $m(0.5)$.



   c. Suppose the medicine is administered at noon. Use the graph to estimate the amount of medicine in the body at 4:30 p.m. on the same day.

4. The area covered by a lake is 11 square kilometers. It is decreasing exponentially at a rate of 2 percent each year and can be modeled by $A(t) = 11 \cdot (0.98)^t$.

   a. By what factor does the area decrease in 10 years?

   b. By what factor does the area decrease each month?

5. The third and fourth numbers in an exponential sequence are 100 and 500. What are the first and second numbers in this sequence?

(From Unit 4, Lesson 1.)

6. The population of a city in thousands is modeled by the function $f(t) = 250 \cdot (1.01)^t$ where t is the number of years after 1950. Which of the following are predicted by the model? Select **all** that apply.

   A. The population in 1950 was 250.

   B. The population in 1950 was 250,000.

   C. The population grows by 1 percent each year.

   D. The population in 1951 was 275,000.

   E. The population grows exponentially.

   (From Unit 4, Lesson 2.)

Lesson 4: Representing Functions at Rational Inputs

- Let's find how quantities are growing or decaying over fractional intervals of time.

4.1: Math Talk: Unknown Exponents

Solve each equation mentally.

1. $5^q = 125$

2. $\frac{1}{5^r} = \frac{1}{125}$

3. $5^t = \frac{1}{125}$

4. $125^u = 5$

4.2: Population of Nigeria

In 1990, Nigeria had a population of about 95.3 million. By 2000, there were about 122.4 million people, an increase of about 28.4%. During that decade, the population can be reasonably modeled by an exponential function.

1. Express the population of Nigeria $f(d)$, in millions of people, d decades since 1990.

2. Write an expression to represent the population of Nigeria in 1996.

3. A student said, "The population of Nigeria grew at a rate of 2.84% every year."
    a. Explain or show why the student's statement is incorrect.

    b. Find the correct annual growth rate. Explain or show your reasoning.

4.3: Got Caffeine?

In healthy adults, caffeine has an average half-life of about 6 hours. Let's suppose a healthy man consumes a cup of coffee that contains 100 mg of caffeine at noon.

1. Each of the following expressions describes the amount of caffeine in the man's body some number of hours after consumption. How many hours after consumption?
    a. $100 \cdot \left(\frac{1}{2}\right)^1$
    b. $100 \cdot \left(\frac{1}{2}\right)^3$
    c. $100 \cdot \left(\frac{1}{2}\right)^{\frac{1}{6}}$
    d. $100 \cdot \left(\frac{1}{2}\right)^t$
2. a. Write a function g to represent the amount of caffeine left in the body, h hours after it enters the bloodstream.

    b. The function f represents the amount of caffeine left in the body after t 6-hour periods. Explain why $g(6) = f(1)$.

Are you ready for more?

What percentage of the initial amount of caffeine do you expect to break down in the first 3 hours: less than 25%, exactly 25%, or more than 25%? Explain or show your reasoning.

Lesson 4 Summary

Imagine a medicine has a half-life of 3 hours. If a patient takes 200 mg of the medicine, then the amount of medicine in their body, in mg, can be modeled by the function $f(t) = 200 \cdot \left(\frac{1}{2}\right)^t$. In this model, t represents a unit of time. Notice that the 200 represents the initial dose the patient took. The number $\frac{1}{2}$ indicates that for every 1 unit of time, the amount of medicine is cut in half. Because the half-life is 3 hours, this means that t must measure time in groups of 3 hours.

But what if we wanted to find the amount of medicine in the patient's body each hour after taking it? We know there are 3 equal groups of 1 hour in a 3-hour period. We also know that because the medicine decays exponentially, it decays by the same factor in each of those intervals. In other words, if b is the decay factor for each hour, then $b \cdot b \cdot b = \frac{1}{2}$ or $b^3 = \frac{1}{2}$. This means that over each hour, the medicine must decay by a factor of $\sqrt[3]{\frac{1}{2}}$, which can also be written as $\left(\frac{1}{2}\right)^{\frac{1}{3}}$. So if h is time in hours since the patient took the medicine, we can express the amount of medicine in mg, g, in the person's body as $g(h) = 200 \cdot \left(\sqrt[3]{\frac{1}{2}}\right)^h$ or $g(h) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{h}{3}}$.

Lesson 4 Practice Problems

1. A bacteria population is tripling every hour. By what factor does the population change in $\frac{1}{2}$ hour? Select **all** that apply.

   A. $\sqrt{3}$

   B. $\frac{3}{2}$

   C. $\sqrt[3]{2}$

   D. $3^{\frac{1}{2}}$

   E. 3^2

2. A medication has a half-life of 4 hours after it enters the bloodstream. A nurse administers a dose of 225 milligrams to a patient at noon.

   a. Write an expression to represent the amount of medication, in milligrams, in the patient's body at:
      i. 1 p.m. on the same day

      ii. 7 p.m. on the same day

   b. The expression $225 \cdot \left(\frac{1}{2}\right)^{\frac{5}{2}}$ represents the amount of medicine in the body some time after it is administered. What is that time?

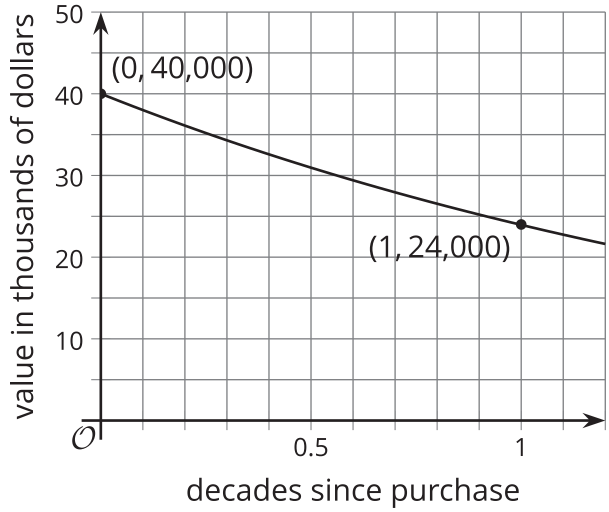
3. The number of employees in a company has been growing exponentially by 10% each year. By what factor does the number of employees change:

   a. Each month?

   b. Every 3 months?

   c. Every 20 months?

4. The value of a truck decreases exponentially since its purchase. The two points on the graph shows the truck's initial value and its value a decade afterward.



a. Express the car's value, in dollars, as a function of time d, in decades, since purchase.

b. Write an expression to represent the car's value 4 years after purchase.

c. By what factor is the value of the car changing each year? Show your reasoning.

5. The value of a stock increases by 8% each year.

a. Explain why the stock value does not increase by 80% each decade.

b. Does the value increase by more or less than 80% each decade?

6. Decide if each statement is true or false.

   a. $50^{\frac{1}{2}} = 25$

   b. $\sqrt{30}$ is a solution to $y^2 = 30$.

   c. $243^{\frac{1}{3}}$ is equivalent to $\sqrt[3]{243}$.

   d. $\sqrt{20}$ is a solution to $m^4 = 20$.

   (From Unit 4, Lesson 3.)

7. Lin is saving $300 per year in an account that pays 4.5% interest per year, compounded annually. About how much money will she have 20 years after she started?

   A. $545.45

   B. $3,748.78

   C. $9,411.43

   D. $1,124,634.54

   (From Unit 2, Lesson 26.)