Lesson 17: Graphs of Rational Functions (Part 1)

- Let's explore graphs and equations of rational functions.

17.1: Biking 10 Miles (Part 1)

Kiran's aunt plans to bike 10 miles.

1. How long will it take if she bikes at an average rate of 8 miles per hour?

2. How long will it take if she bikes at an average rate of r miles per hour?

3. Kiran wants to join his aunt, but he only has 45 minutes to exercise. What will their average rate need to be for him to finish on time?

4. What will their average rate need to be if they have t hours to exercise?

17.2: Biking 10 Miles (Part 2)

Kiran plans to bike 10 miles.

1. Write an equation that gives his time t, in hours, as a function of his rate r, in miles per hour.

2. Graph $y = t(r)$.

3. What is the meaning of $t(8)$? Does this value make sense? Explain your reasoning.

4. What is the meaning of $t(0)$? Does this value make sense? Explain your reasoning.

5. As r gets closer and closer to 0, what does the behavior of the function tell you about the situation?

6. As r gets larger and larger, what does the end behavior of the function tell you about the situation?

17.3: Card Sort: Graphs of Rational Functions

Your teacher will give you a set of cards. Match each rational function with its graphical representation.

Are you ready for more?

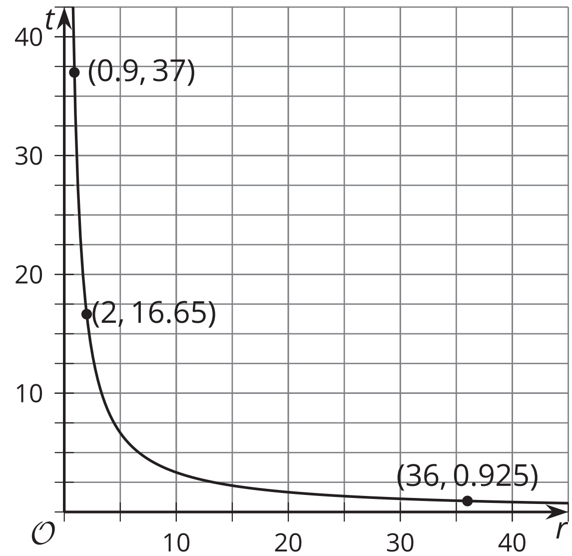
Priya and Han are bicycling. Han is going at a rate of 10 mph and begins 2 miles ahead of Priya. If Priya bikes at a rate of r mph, when will Priya pass Han? Write an equation and sketch a graph. Then interpret the graph in terms of the situation.

Lesson 17 Summary

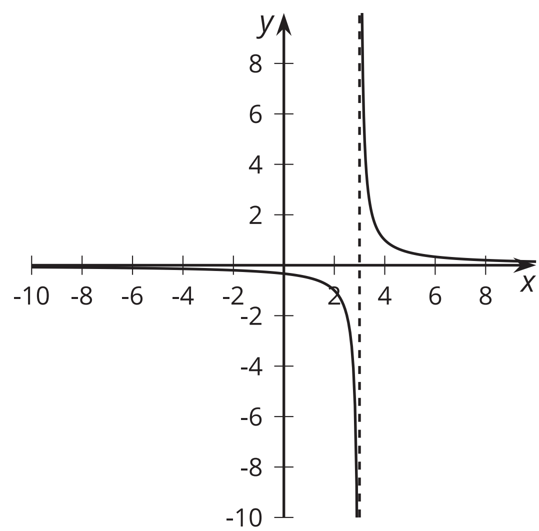
The distance d that an object moving at constant speed travels is based on the length of time t the object travels and the speed r of the object. Often, this relationship is written as $d = r \cdot t$. We could also write the relationship as $r = \frac{d}{t}$ or $t = \frac{d}{r}$. Depending on what we want to know, one form of this relationship may be more useful than another.

For example, the distance across the English Channel from Dover in England to Calais in France is 33.3 km. The time in hours it takes for a boat to make this crossing can be modeled by the function $T(r) = \frac{33.3}{r}$, where r is measured in kilometers per hour.

For very small values of r, the journey takes a long time. For larger values of r (and a fast boat!), the trip is shorter. The graph of the function shows how the travel time decreases as the speed of the boat increases.



Unlike the graphs of polynomial functions that look smooth and connected, the graphs of some rational functions can look like separate pieces. For example, here is a graph of $f(x) = \frac{1}{x-3}$.



The dashed line at $x = 3$ is a representation of a **vertical asymptote**. As x gets closer and closer to 3, think about what happens to the value of the expression for $f(x)$. If we divide 1 by a very small negative number, we get a very big negative number, which is what happens on the left side of the vertical asymptote. If we divide 1 by a very small positive number, we get a very big positive number, which is what happens on the right side of the vertical asymptote. It is important to note that the drawn-in asymptote is not actually part of the graph of the function. Instead, it is a helpful reminder that the function has no value at $x = 3$ and very large absolute values at inputs very close to $x = 3$.

Glossary

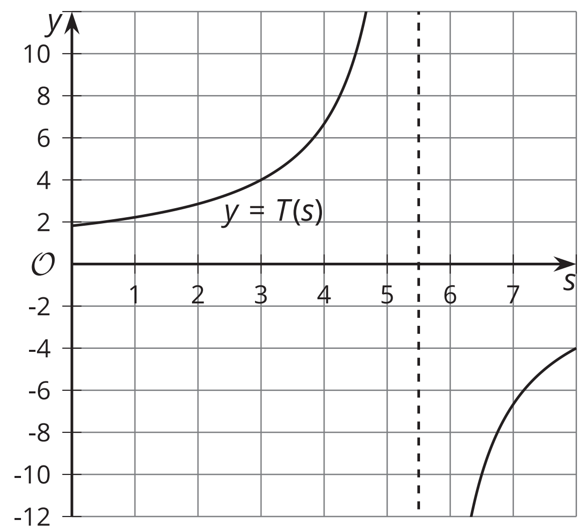
- vertical asymptote

Lesson 17 Practice Problems

1. Jada is planning a kayak trip. She finds an expression for the time, $T(s)$, in hours it takes her to paddle 10 kilometers upstream in terms of s, the speed of the current in kilometers per hour. This is the graph Jada gets if she allows s to take on any value between 0 and 7.5.

   a. What would be a more appropriate domain for Jada to use instead?

   b. What is the approximate speed of the current if her trip takes 6 hours?



2. A cylindrical can needs to have a volume of 6 cubic inches. A label is to go around the side of the can. The function $S(r) = \frac{12}{r}$ gives the area of the label in square inches where r is the radius of the can in inches.

   a. As r gets closer and closer to 0, what does the behavior of the function tell you about the situation?

   b. As r gets larger and larger, what does the end behavior of the function tell you about the situation?

3. What is the equation of the vertical asymptote for the graph of the rational function $g(x) = \frac{6}{x-1}$?

   A. $x = 1$

   B. $x = -1$

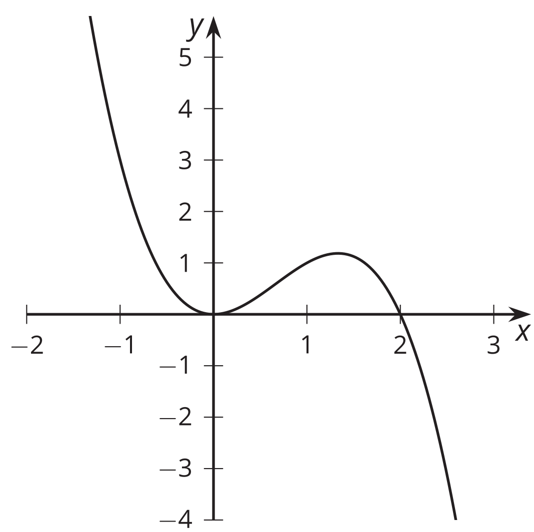
   C. $x = 6$

   D. $x = \frac{1}{6}$

4. A geometric sequence h starts at 16 and has a growth factor of 1.75. Sketch a graph of h showing the first 5 terms.

   (From Unit 1, Lesson 7.)

5. Is this the graph of $g(x) = -x^2(x-2)$ or $h(x) = x^2(x-2)$? Explain how you know.



   (From Unit 2, Lesson 10.)

6. *Technology required*. A 6 oz cylindrical can of tomato paste needs to have a volume of 178 cm^3. The current can design uses a radius of 2.75 cm and a height of 7.5 cm. Use graphing technology to find a cylindrical design that would have less surface area so each can uses less metal.

   (From Unit 2, Lesson 16.)

7. The surface area $S(r)$ in square units of a cylinder with a volume of 20 cubic units is a function of its radius r in units where $S(r) = 2\pi r^2 + \frac{40}{r}$. What is the surface area of a cylinder with a volume of 20 cubic units and a radius of 4 units?

   (From Unit 2, Lesson 16.)

Lesson 18: Graphs of Rational Functions (Part 2)

- Let's learn about horizontal asymptotes.

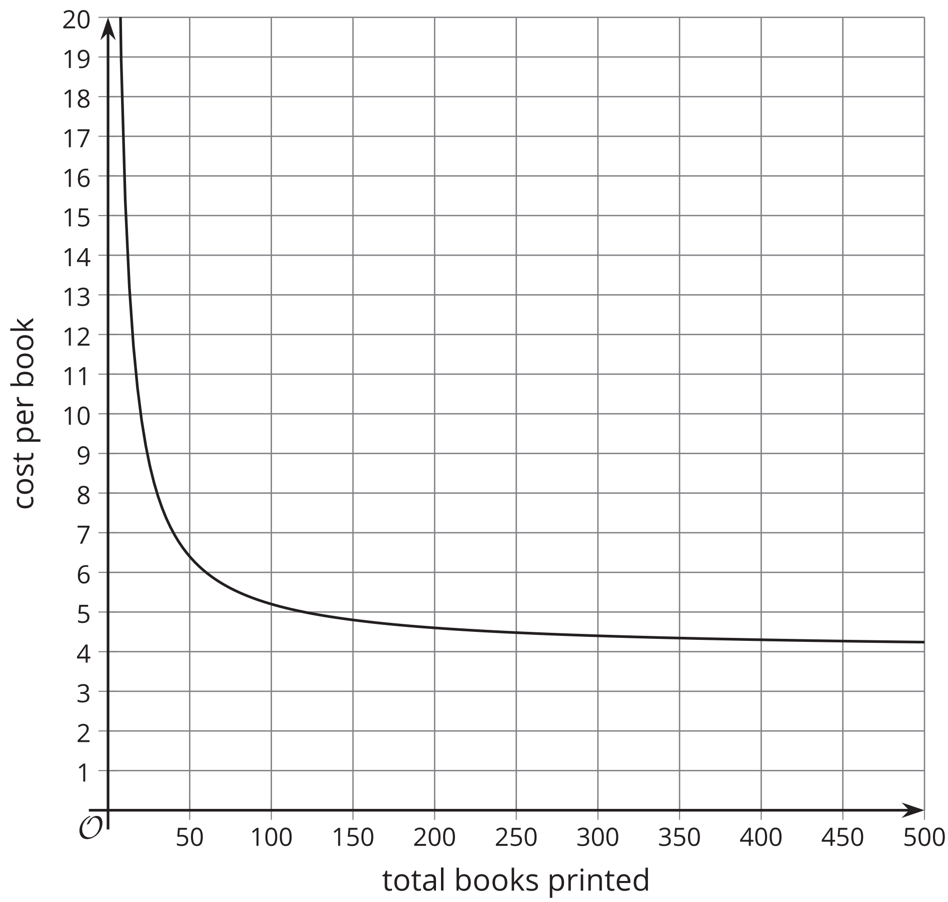
18.1: Rewritten Equations

Decide if each of these equations is true or false for x values that do not result in a denominator of 0. Be prepared to explain your reasoning.

1. $\frac{x+7}{x} = 1 + \frac{7}{x}$
2. $\frac{x}{x+7} = 1 + \frac{x}{7}$

18.2: Publishing a Paperback

Let c be the function that gives the average cost per book $c(x)$, in dollars, when using an online store to print x copies of a self-published paperback book. Here is a graph of $c(x) = \frac{120+4x}{x}$.



1. What is the approximate cost per book when 50 books are printed? 100 books?

2. The author plans to charge $8 per book. About how many should be printed to make a profit?

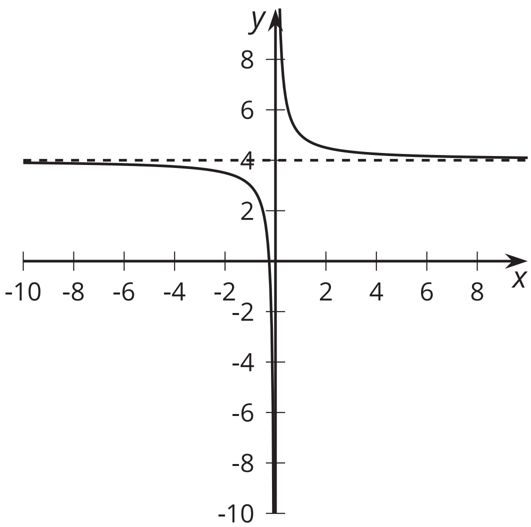
3. What is the value of $c(x)$ when $x = \frac{1}{2}$? How does this relate to the context?

4. What does the end behavior of the function say about the context?

18.3: Horizontal Asymptotes

Here are four graphs of rational functions.

A



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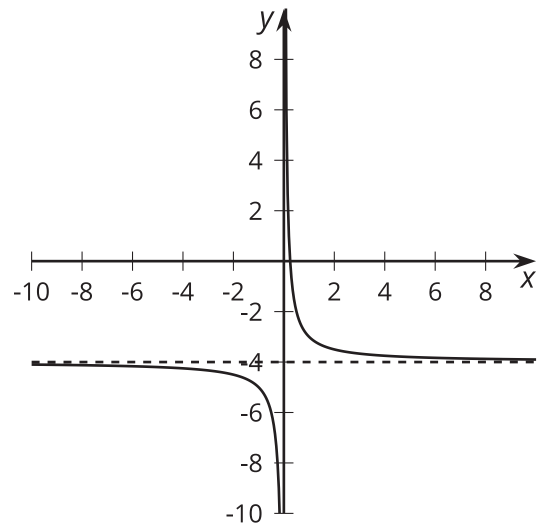

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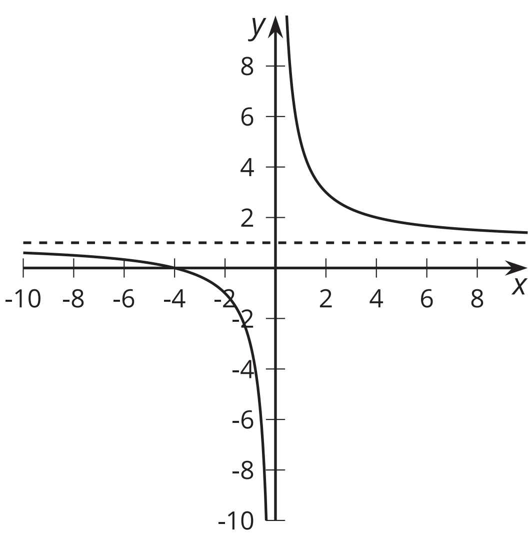
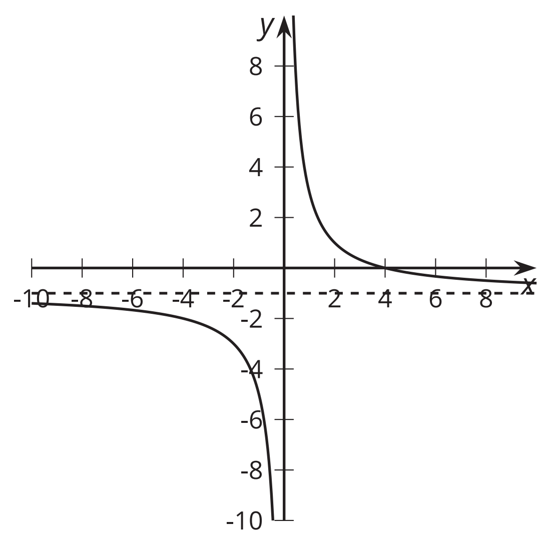


D



1. Match each function with its graphical representation.
   a. $a(x) = \frac{4}{x} - 1$

   b. $b(x) = \frac{1}{x} - 4$

   c. $c(x) = \frac{1+4x}{x}$

   d. $d(x) = \frac{x+4}{x}$

e. $e(x) = \frac{1-4x}{x}$

f. $f(x) = \frac{4-x}{x}$

g. $g(x) = 1 + \frac{4}{x}$

h. $h(x) = \frac{1}{x} + 4$

2. Where do you see the **horizontal asymptote** of the graph in the expressions for the functions?

Are you ready for more?

Consider the function $a(x) = \frac{\frac{1}{2}x+1}{x-1}$.

1. Predict where you think the vertical and horizontal asymptotes of $a(x)$ will be. Explain your reasoning.

2. Use graphing technology to check your prediction.

Lesson 18 Summary

Consider the rational function $f(x) = \frac{3x+1}{x}$. Written this way, we can tell that the graph of the function has a vertical asymptote at $x = 0$ by reading the denominator and identifying the value that would cause division by zero. But what can we tell about the value of $f(x)$ for values of x far away from the vertical asymptote?

One way we can think about these values is to rewrite the expression for $f(x)$ by breaking up the fraction:

$$f(x) = \frac{3x}{x} + \frac{1}{x}$$
$$f(x) = 3 + \frac{1}{x}$$

Written this way, it's easier to see that as x gets larger and larger in either the positive or negative direction, the $\frac{1}{x}$ term will get closer and closer to 0. Because of this, we can say that the value of the function will get closer and closer to 3.

More generally, if a rational function $g(x) = \frac{a(x)}{b(x)}$ can be rewritten as $g(x) = c + \frac{r(x)}{b(x)}$, where c is a constant, and $r(x)$ and $b(x)$ are polynomial expressions where $\frac{r(x)}{b(x)}$ gets closer and closer to zero as x gets larger and larger in both the positive and negative directions, then $g(x)$ will get closer and closer to c.

Rational functions of this type have a **horizontal asymptote** at the constant value. The line $y = c$ is a horizontal asymptote for f if $f(x)$ gets closer and closer to c as the magnitude of x increases.

Glossary

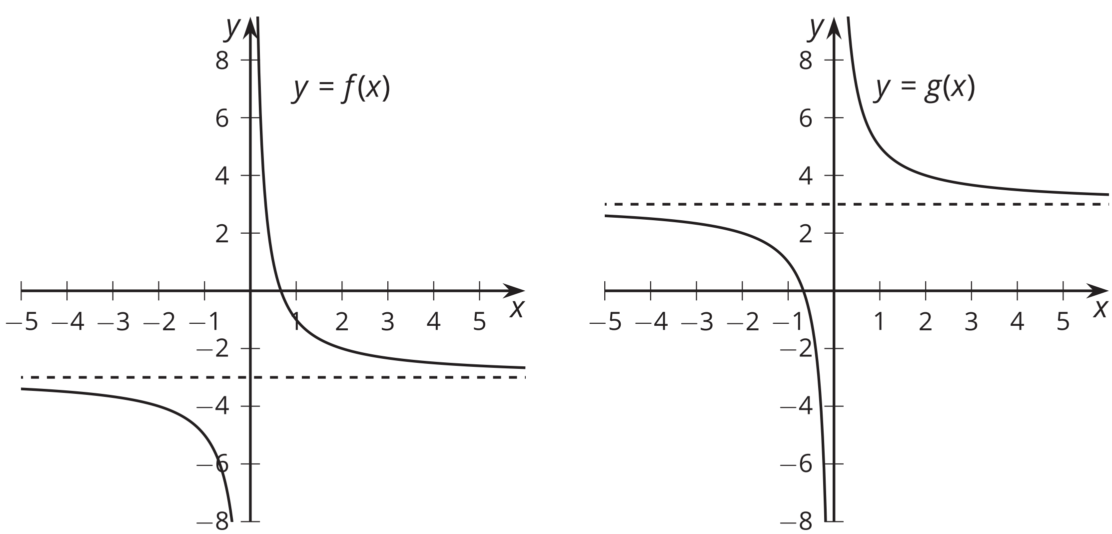
- horizontal asymptote

Lesson 18 Practice Problems

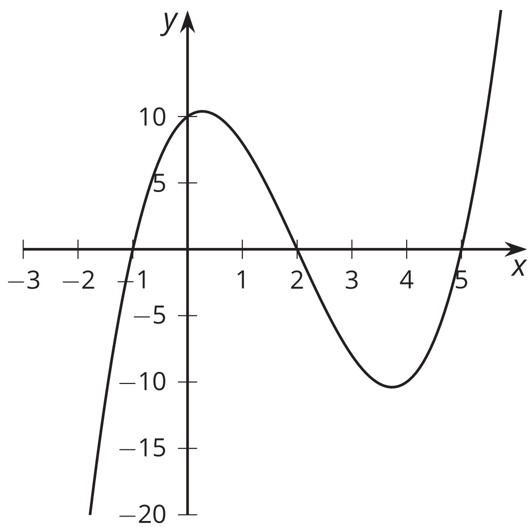
1. Rewrite the rational function $g(x) = \frac{x-4}{x}$ in the form $g(x) = c + \frac{r}{x}$, where c and r are constants.

2. The average cost (in dollars) per mile for riding x miles in a cab is $c(x) = \frac{2.5+2x}{x}$. As x gets larger and larger, what does the end behavior of the function tell you about the situation?

3. The graphs of two rational functions f and g are shown. One of them is given by the expression $\frac{2-3x}{x}$. Which graph is it? Explain how you know.



4. Which polynomial function's graph is shown here?



A. $f(x) = (x+1)(x+2)(x+5)$

B. $f(x) = (x+1)(x-2)(x-5)$

C. $f(x) = (x-1)(x+2)(x+5)$

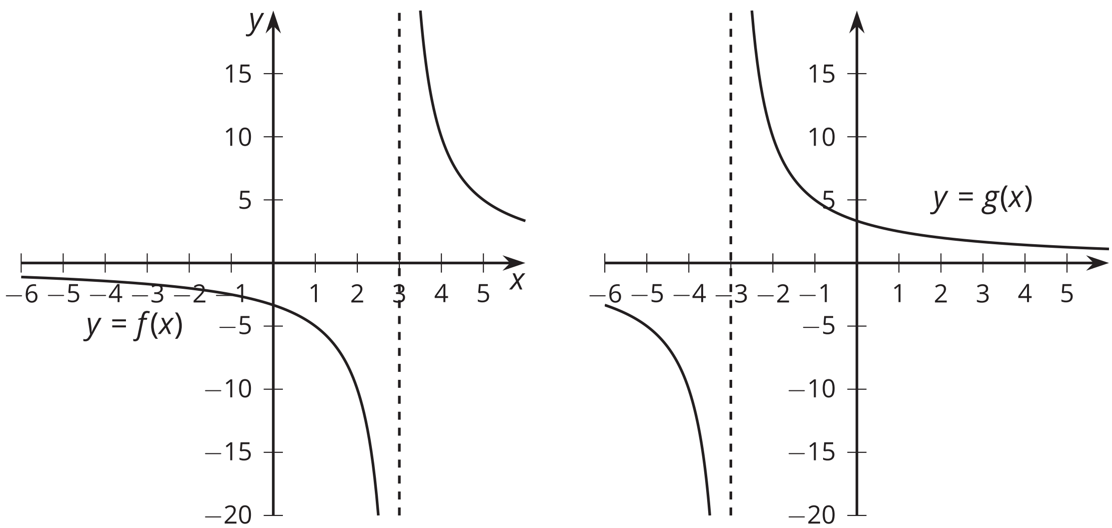
D. $f(x) = (x-1)(x-2)(x-5)$

(From Unit 2, Lesson 7.)

5. State the degree and end behavior of $f(x) = 5x^3 - 2x^4 - 6x^2 - 3x + 7$. Explain or show your reasoning.

(From Unit 2, Lesson 9.)

6. The graphs of two rational functions f and g are shown. Which function must be given by the expression of $\frac{10}{x-3}$? Explain how you know.



(From Unit 2, Lesson 17.)