Lesson 14: What Do You Know About Polynomials?

- Let's put together what we've learned about polynomials so far.

14.1: What Else is True?

$G(x)$ is a polynomial. Here are some things we know about it:

- It has degree 3.
- Both x and $(x + 4)$ are factors of G.
- It has 2 horizontal intercepts, but only 1 is negative.
- Its leading coefficient is negative.

What else do we know is true about $G(x)$?

14.2: Info Gap: More Polynomials

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the data card:

1. Silently read the information on your card.
2. Ask your partner, “What specific information do you need?” and wait for your partner to ask for information. Only give information that is on your card. (Do not figure out anything for your partner!)
3. Before telling your partner the information, ask, “Why do you need to know (that piece of information)?”
4. Read the problem card, and solve the problem independently.
5. Share the data card, and discuss your reasoning.

If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card, and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

14.3: Even More Polynomials

1. Without letting your partner see, do the following:
    a. write a polynomial of degree 3 or 4 in factored form

    b. sketch the graph of your polynomial

    c. rewrite its expression in standard form

2. On a separate slip of paper, write the standard form of your polynomial along with 1 of the factors (or 2 factors, if the polynomial has degree 4). Trade slips with your partner.

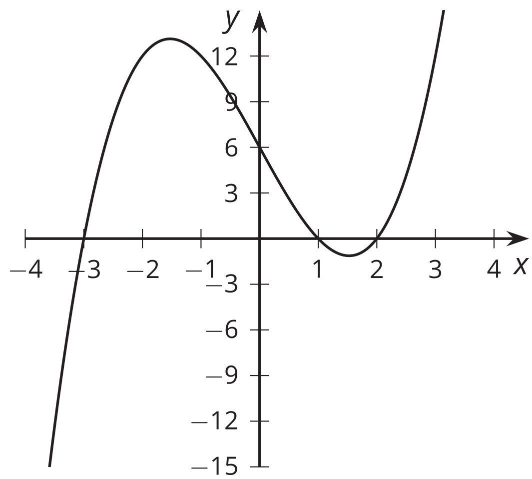
3. Use the information your partner gave you about their polynomial to:
    a. rewrite their polynomial in factored form

    b. sketch a graph of their polynomial showing all horizontal intercepts

4. Once you and your partner have finished graphing, check your factored form and graph with your partner and discuss any differences.

Lesson 14 Summary

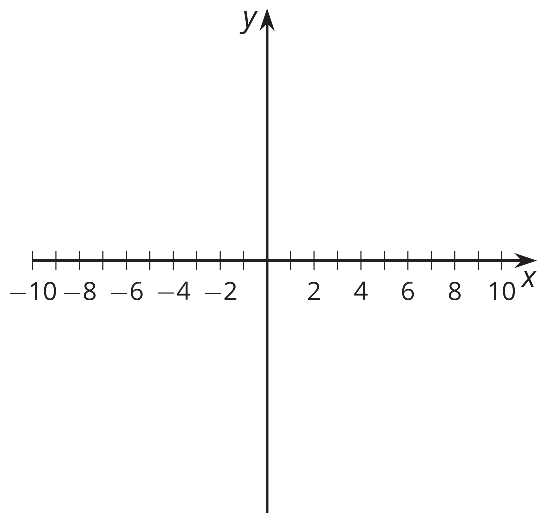
We can look at the same polynomial in many different ways. Let's think about $P(x) = x^3 - 7x + 6$. It's written in standard form, but we could also write it in factored form as $(x-2)(x+3)(x-1)$. If we graph $P(x)$, we get this:



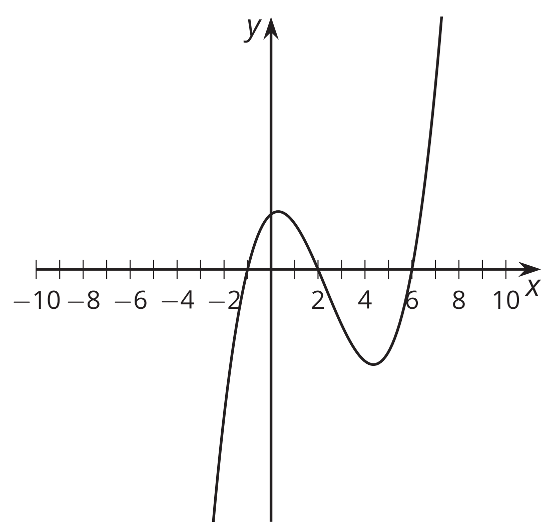
Depending on what we know about $P(x)$ and what we want to do, different forms of it will be more useful. If we want to quickly estimate the value of $P(x)$ for some value of x, the graph might be most helpful. If we don't know what the graph of $P(x)$ looks like, the factored form can help us find the zeros and sketch it. If we want to know the general shape of the graph, we can use the standard form to find the end behavior. If we want to know the factors of $P(x)$ and we only know the standard form, we can guess some possible factors and divide $P(x)$ by them. If we have the factored form and we want to know the standard form, we can multiply all the factors together.

Lesson 14 Practice Problems

1. We know these things about a polynomial function, $f(x)$: it has exactly one relative maximum and one relative minimum, it has exactly three zeros, and it has a known factor of $(x - 4)$. Sketch a graph of $f(x)$ given this information.

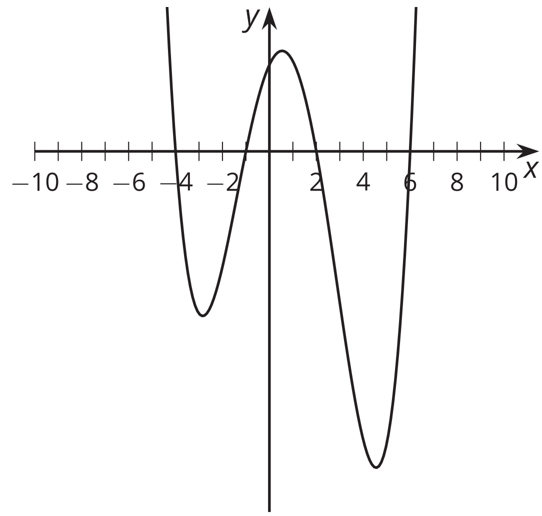


2. Mai graphs a polynomial function, $f(x)$, that has three linear factors $(x + 6)$, $(x + 2)$, and $(x - 1)$. But she makes a mistake. What is her mistake?



3. Here is the graph of a polynomial function with degree 4.

   Select **all** of the statements that are true about the function.



   A. The leading coefficient is positive.

   B. The constant term is negative.

   C. It has 2 relative maximums.

   D. It has 4 linear factors.

   E. One of the factors is $(x-1)$.

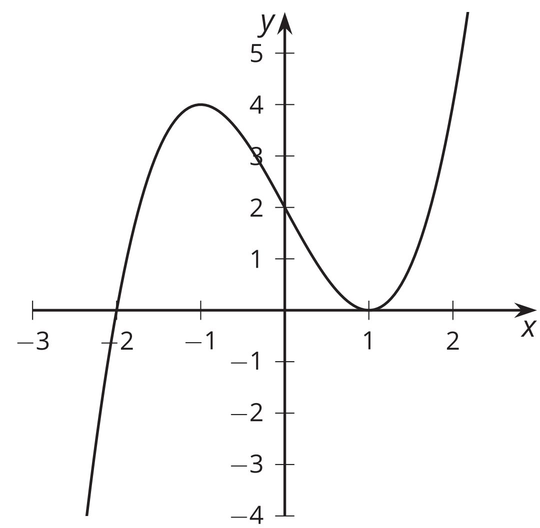
   F. One of the zeros is $x=2$.

   G. There is a relative minimum between $x=1$ and $x=3$.

4. State the degree and end behavior of $f(x)=2x^3-3x^5-x^2+1$. Explain or show your reasoning.

(From Unit 2, Lesson 9.)

5. Is this the graph of $g(x) = (x-1)^2(x+2)$ or $h(x) = (x-1)(x+2)^2$? Explain how you know.



(From Unit 2, Lesson 10.)

6. Kiran thinks he knows one of the linear factors of $P(x) = x^3 + x^2 - 17x + 15$. After finding that $P(3) = 0$, Kiran suspects that $x - 3$ is a factor of $P(x)$, so he sets up a diagram to check. Here is the diagram he made to check his reasoning, but he set it up incorrectly. What went wrong?

	x^2	$4x$	**-5**
x	x^3	$4x^2$	$-5x$
3	$3x^2$	$12x$	15

(From Unit 2, Lesson 12.)

7. The polynomial function $B(x) = x^3 + 8x^2 + 5x - 14$ has a known factor of $(x+2)$. Rewrite $B(x)$ as a product of linear factors.

(From Unit 2, Lesson 13.)

Lesson 15: The Remainder Theorem

- Let’s learn about the Remainder Theorem.

15.1: Notice and Wonder: Division Leftovers

What do you notice? What do you wonder?

33	82	66
10) 330	4) 330	5) 330
300	320	300
30	10	30
30	8	30
0	2	0

A. $330 = 33(10) + 0$

B. $330 = 4(82) + 2$

C. $330 = 5(66) + 0$

15.2: The Unknown Coefficient

Consider the polynomial function $f(x) = x^4 - ux^3 + 24x^2 - 32x + 16$ where u is an unknown real number. If $x - 2$ is a factor, what is the value of u? Explain how you know.

Are you ready for more?

Here are some diagrams that show the same third-degree polynomial, $P(x) = 2x^3 + 5x^2 + x + 10$, divided by a linear factor and by a quadratic factor.

$$\frac{P(x)}{x+3}$$

	$2x^2$	$-x$	4
x	$2x^3$	$-x^2$	$4x$
3	$6x^2$	$-3x$	12

$$\frac{P(x)}{x^2 - x}$$

	$2x$	7
x^2	$2x^3$	$7x^2$
$-x$	$-2x^2$	$-7x$

1. What is the remainder of each of these divisions?

2. For each division, how does the degree of the remainder compare to the degree of the divisor?

3. Could the remainder ever have the same degree as the divisor, or a higher degree? Give an example to show that this is possible, or explain why it is not possible.

15.3: A Study of Remainders

1. Which of these polynomials could have $(x - 2)$ as a factor?
   a. $A(x) = 6x^2 - 7x - 5$

   b. $B(x) = 3x^2 + 15x - 42$

   c. $C(x) = 2x^3 + 13x^2 + 16x + 5$

   d. $D(x) = 3x^3 - 2x^2 - 15x + 14$

   e. $E(x) = 8x^4 - 41x^3 - 18x^2 + 101x + 70$

   f. $F(x) = x^4 + 5x^3 - 27x^2 - 101x - 70$

2. Select one of the polynomials that you said doesn't have $(x - 2)$ as a factor.
   a. Explain how you know $(x - 2)$ is not a factor.

   b. If you have not already done so, divide the polynomial by $(x - 2)$. What is the remainder?

3. List the remainders for each of the polynomials when divided by $(x - 2)$. How do these values compare to the value of the functions at $x = 2$?

Lesson 15 Summary

When we use long division to divide 1573 by 12, we get a remainder of 1, so $1573 = 12(131) + 1$. When we divide by 11 instead, we get a remainder of 0, so $1573 = 11(143)$. A remainder of 0 means that 11 is a factor of 1573. The same thing happens with polynomials. While $(x^3 + 5x^2 + 7x + 3) \div (x + 2)$ results in a remainder that is not 0, if we divide $(x + 1)$ into $x^3 + 5x^2 + 7x + 3$, we do get a remainder of 0:

$$
\begin{array}{r}
x^2 + 4x + 3 \\
x + 1 \overline{) x^3 + 5x^2 + 7x + 3} \\
\underline{-x^3 - x^2} \\
4x^2 + 7x \\
\underline{-4x^2 - 4x} \\
3x + 3 \\
\underline{-3x - 3} \\
0
\end{array}
$$

So $(x + 1)$ is a factor of $x^3 + 5x^2 + 7x + 3$.

Earlier we learned that if $(x - a)$ is a factor of a polynomial $p(x)$, then $p(a) = 0$, meaning a is a zero of the function. It turns out that the converse is also true: if a is a zero, then $(x - a)$ is a factor.

To see that this is true, let's think about what we know if we have a polynomial $p(x)$ with a known zero at $x = a$. If we divide $p(x)$ by the linear factor $(x - a)$, then $p(x) = (x - a)q(x) + r$, where r is the remainder and $q(x)$ is a polynomial. Because a is a zero of the function, we know that $p(a) = 0$. This means we also know that the remainder is zero:

$p(a) = (a - a)q(x) + r$
$p(a) = r$
$0 = r$

Which means that $p(x) = (x - a)q(x)$. So, if a is a zero of a polynomial, then $(x - a)$ must be a factor of $p(x)$. Now we know that if we start with a linear factor of a polynomial, then we know one of the zeros of the polynomials, and if we start with a zero of a polynomial, then we know one of the linear factors.

Lastly, even if a is not a zero of p, we can figure out what the remainder will be if we divide $p(x)$ by $(x - a)$, without having to do any division. If $p(x) = (x - a)q(x) + r$, then $p(a) = (a - a)q(x) + r$, so $p(a) = r$. So the remainder after division by $(x - a)$ is $p(a)$. This is the Remainder Theorem.

Lesson 15 Practice Problems

1. For the polynomial function $f(x) = x^3 - 2x^2 - 5x + 6$, we have $f(0) = 6$, $f(2) = -4$, $f(-2) = 0$, $f(3) = 0$, $f(-1) = 8$, $f(1) = 0$. Rewrite $f(x)$ as a product of linear factors.

2. Select **all** the polynomials that have $(x - 4)$ as a factor.

   A. $x^3 - 13x - 12$

   B. $x^3 + 8x^2 + 19x + 12$

   C. $x^3 + 6x + 5x - 12$

   D. $x^3 - x^2 - 10x - 8$

   E. $x^2 - 4$

3. Write a polynomial function, $p(x)$, with degree 3 that has $p(7) = 0$.

4. Long division was used here to divide the polynomial function $p(x) = x^3 + 7x^2 - 20x - 110$ by $(x - 5)$ and to divide it by $(x + 5)$.

$$\begin{array}{r}
x^2 + 12x + 40 \\
x - 5 \overline{) x^3 + 7x^2 - 20x - 110} \\
\underline{-x^3 + 5x^2} \qquad\qquad\quad \\
12x^2 - 20x \qquad\quad \\
\underline{-12x^2 + 60x} \qquad\quad \\
40x - 110 \\
\underline{-40x + 200} \\
90
\end{array}
\qquad\qquad
\begin{array}{r}
x^2 + 2x - 30 \\
x + 5 \overline{) x^3 + 7x^2 - 20x - 110} \\
\underline{-x^3 - 5x^2} \qquad\qquad\quad \\
2x^2 - 20x \qquad\quad \\
\underline{-2x^2 - 10x} \qquad\quad \\
-30x - 110 \\
\underline{30x + 150} \\
40
\end{array}$$

a. What is $p(-5)$?

b. What is $p(5)$?

5. Which polynomial function has zeros when $x = 5, \frac{2}{3}, -7$?

A. $f(x) = (x + 5)(2x + 3)(x - 7)$

B. $f(x) = (x + 5)(3x + 2)(x - 7)$

C. $f(x) = (x - 5)(2x - 3)(x + 7)$

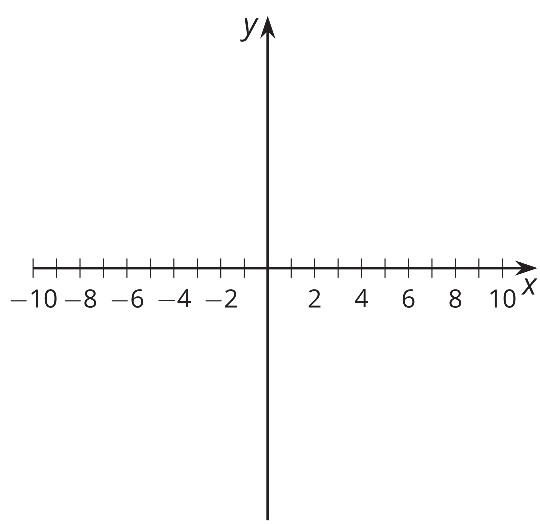
D. $f(x) = (x - 5)(3x - 2)(x + 7)$

(From Unit 2, Lesson 5.)

6. The polynomial function $q(x) = 3x^4 + 8x^3 - 13x^2 - 22x + 24$ has known factors $(x + 3)$ and $(x + 2)$. Rewrite $q(x)$ as the product of linear factors.

(From Unit 2, Lesson 12.)

7. We know these things about a polynomial function $f(x)$: it has degree 3, the leading coefficient is negative, and it has zeros at $x =$ -5, -1, 3. Sketch a graph of $f(x)$ given this information.



(From Unit 2, Lesson 14.)