Lesson 12: Polynomial Division (Part 1)

- Let's learn a way to divide polynomials.

12.1: Notice and Wonder: A Different Use for Diagrams

What do you notice? What do you wonder?

A. $(x-3)(x+5)=x^2+2x-15$

	x	5
x	x^2	$5x$
-3	$-3x$	-15

B. $(x-1)(x^2+3x-4)=x^3+2x^2-7x+4$

	x^2	$3x$	-4
x	x^3	$3x^2$	$-4x$
-1	$-x^2$	$-3x$	+4

C. $(x-2)(?)=(x^3-x^2-4x+4)$

x	x^3		
-2			

12.2: Factoring with Diagrams

Priya wants to sketch a graph of the polynomial f defined by $f(x) = x^3 + 5x^2 + 2x - 8$. She knows $f(1) = 0$, so she suspects that $(x - 1)$ could be a factor of $x^3 + 5x^2 + 2x - 8$ and writes $(x^3 + 5x^2 + 2x - 8) = (x - 1)(?x^2 + ?x + ?)$ and draws a diagram.

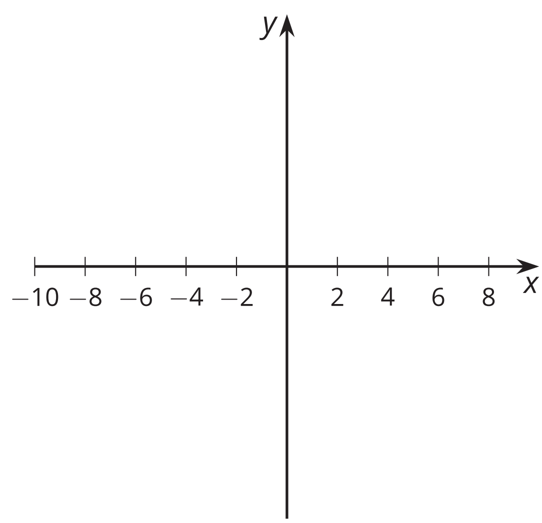
x	x^3		
-1			

1. Finish Priya's diagram.

2. Write $f(x)$ as the product of $(x - 1)$ and another factor.

3. Write $f(x)$ as the product of three linear factors.

4. Make a sketch of $y = f(x)$.



12.3: More Factoring with Diagrams

Here are some polynomial functions with known factors. Rewrite each polynomial as a product of linear factors. Note: you may not need to use all the columns in each diagram. For some problems, you may need to make another diagram.

1. $A(x) = x^3 - 7x^2 - 16x + 112$, $(x - 7)$

	x^2				
x	x^3	0			
-7	$-7x^2$				

2. $B(x) = 2x^3 - x^2 - 27x + 36$, $\left(x - \frac{3}{2}\right)$

	$2x^2$				
x	$2x^3$	$2x^2$			
$-\frac{3}{2}$	$-3x^2$				

3. $C(x) = x^3 - 3x^2 - 13x + 15$, $(x + 3)$

x					
3					

4. $D(x) = x^4 - 13x^2 + 36$, $(x - 2)$, $(x + 2)$

(Hint: $x^4 - 13x^2 + 36 = x^4 + 0x^3 - 13x^2 + 0x + 36$)

5. $F(x) = 4x^4 - 15x^3 - 48x^2 + 109x + 30$, $(x - 5)$, $(x - 2)$, $(x + 3)$

Are you ready for more?

A diagram can also be used to divide polynomials even when a factor is not linear. Suppose we know $(x^2 - 2x + 5)$ is a factor of $x^4 + x^3 - 5x^2 + 23x - 20$. We could write $(x^4 + x^3 - 5x^2 + 23x - 20) = (x^2 - 2x + 5)(?x^2 + ?x + ?)$. Make a diagram and find the missing factor.

Lesson 12 Summary

What are some things that could be true about the polynomial function defined by $p(x) = x^3 - 5x^2 - 2x + 24$ if we know $p(\text{-}2) = 0$? If we think about the graph of the polynomial, the point $(\text{-}2, 0)$ must be on the graph as a horizontal intercept. If we think about the expression written in factored form, $(x + 2)$ could be one of the factors, since $x + 2 = 0$ when $x = \text{-}2$. How can we figure out whether $(x + 2)$ actually is a factor?

Well, if we assume $(x + 2)$ is a factor, there is some other polynomial $q(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $p(x) = (x + 2)q(x)$. (Can you see why $q(x)$ has to have a degree of 2?) In the past, we have done things like expand $(x + 2)(ax^2 + bx + c)$ to find $p(x)$. Since we already know the expression for $p(x)$, we can instead work out the values of a, b, and c by thinking through the calculation.

One way to organize our thinking is to use a diagram. We first fill in $(x + 2)$ and the leading term of $p(x)$, x^3. From this start, we see the leading term of $q(x)$ must be x^2, meaning $a = 1$, since $x \cdot x^2 = x^3$.

	x^2		
x	x^3		
+2			

We then fill in the rest of the diagram using similar thinking and paying close attention to the signs of each term. For example, we put in a $2x^2$ in the bottom left cell because that's the product of 2 and x^2. But that means we need to have a -$7x^2$ in the middle cell of the middle row, since that's the only other place we will get an x^2 term, and we need to get -$5x^2$ once all the terms are collected. Continuing in this way, we get the completed table:

	x^2	-$7x$	**+12**
x	x^3	-$7x^2$	$+12x$
+2	$+2x^2$	-$14x$	+24

Collecting all the terms in the interior of the diagram, we see that $x^3 - 5x^2 - 2x + 24 = (x + 2)(x^2 - 7x + 12)$, so $q(x) = x^2 - 7x + 12$. Notice that the 24 in the bottom right was exactly what we needed, and it's how we know that $(x + 2)$ is a factor of $p(x)$. In a future lesson, we will see why this happened. With a bit more factoring, we can say that $p(x) = (x + 2)(x - 3)(x - 4)$.

Lesson 12 Practice Problems

1. The polynomial function $p(x) = x^3 - 3x^2 - 10x + 24$ has a known factor of $(x - 4)$.

   a. Rewrite $p(x)$ as the product of linear factors.

   b. Draw a rough sketch of the graph of the function.

2. Tyler thinks he knows one of the linear factors of $P(x) = x^3 - 9x^2 + 23x - 15$. After finding that $P(1) = 0$, he suspects that $x - 1$ is a factor of $P(x)$. Here is the diagram he made to check if he's right, but he set it up incorrectly. What went wrong?

	x^2	-8x	-15
x	x^3	-8x^2	-15x
1	x^2	-8x	-15

3. The polynomial function $q(x) = 2x^4 - 9x^3 - 12x^2 + 29x + 30$ has known factors $(x - 2)$ and $(x + 1)$. Which expression represents $q(x)$ as the product of linear factors?

   A. $(2x - 5)(x + 3)(x - 2)(x + 1)$

   B. $(2x + 3)(x - 5)(x - 2)(x + 1)$

   C. $(2x + 15)(x - 1)(x - 2)(x + 1)$

   D. $(2x - 15)(x + 1)(x - 2)(x + 1)$

4. Each year a certain amount of money is deposited in an account which pays an annual interest rate of r so that at the end of each year the balance in the account is multiplied by a growth factor of $x = 1 + r$. \$1,000 is deposited at the start of the first year, an additional \$300 is deposited at the start of the next year, and \$500 at the start of the following year.

   a. Write an expression for the value of the account at the end of three years in terms of the growth factor x.

   b. Determine (to the nearest cent) the amount in the account at the end of three years if the interest rate is 4%.

   (From Unit 2, Lesson 2.)

5. State the degree and end behavior of $f(x) = 5 + 7x - 9x^2 + 4x^3$. Explain or show your reasoning.

   (From Unit 2, Lesson 8.)

6. Describe the end behavior of $f(x) = 1 + 7x + 9x^3 + 6x^4 - 2x^5$.

(From Unit 2, Lesson 10.)

7. What are the points of intersection between the graphs of the functions $f(x) = (x + 3)(x - 1)$ and $g(x) = (x + 1)(x - 3)$?

(From Unit 2, Lesson 11.)

Lesson 13: Polynomial Division (Part 2)

- Let's learn a different way to divide polynomials.

13.1: Notice and Wonder: Different Divisions

What do you notice? What do you wonder?

$$\begin{array}{r} 2 \\ 11\overline{)2772} \\ \underline{22} \\ 5 \end{array} \qquad \begin{array}{r} 25 \\ 11\overline{)2772} \\ \underline{22} \\ 57 \\ \underline{55} \\ 2 \end{array} \qquad \begin{array}{r} 252 \\ 11\overline{)2772} \\ \underline{22} \\ 57 \\ \underline{55} \\ 22 \\ \underline{22} \\ 0 \end{array}$$

$$\begin{array}{r} 2x^2 \\ x+1\overline{)2x^3 + 7x^2 + 7x + 2} \\ \underline{-2x^3 - 2x^2} \\ 5x^2 + 7x \end{array}$$

13.2: Polynomial Long Division

1. Diego used the long division shown here to figure out that $6x^2 - 7x - 5 = (2x+1)(3x-5)$. Show what it would look like if he had used a diagram.

$$\begin{array}{r} 3x - 5 \\ 2x+1\overline{)6x^2 - 7x - 5} \\ \underline{-6x^2 - 3x} \\ -10x - 5 \\ \underline{10x + 5} \\ 0 \end{array}$$

$2x$	$6x^2$	
1		

Pause here for a whole-class discussion.

2. $(x-2)$ is a factor of $2x^3-7x^2+x+10$, which means there is some other factor A where $2x^3-7x^2+x+10=(x-2)(A)$. Finish the division started here to find the value of A.

$$\begin{array}{r} 2x^2 \\ x-2\overline{\smash{)}2x^3-7x^2+x+10} \\ \underline{-2x^3+4x^2} \end{array}$$

3. Jada used the diagram shown here to figure out that $2x^3+13x^2+16x+5=(2x+1)(x^2+6x+5)$. Show what it would look like if she had used long division.

	x^2	$6x$	5
$2x$	$2x^3$	$12x^2$	$10x$
1	x^2	$6x$	5

$$2x+1\overline{\smash{)}2x^3+13x^2+16x+5}$$

Are you ready for more?

1. What is $(x^4-1) \div (x-1)$?

2. Use your response to predict what $(x^7 - 1) \div (x - 1)$ is, and then use division to check your prediction.

13.3: More Long Division

Here are some polynomial functions with known factors. Rewrite each polynomial as a product of linear factors using long division.

1. $A(x) = x^3 - 7x^2 - 16x + 112$, $(x - 7)$

$$\begin{array}{r} x^2 \\ x - 7 \overline{) x^3 - 7x^2 - 16x + 112} \\ \underline{-x^3 + 7x^2} \end{array}$$

2. $C(x) = x^3 - 3x^2 - 13x + 15$, $(x + 3)$

13.4: Missing Numbers

Here are pairs of equivalent expressions, one in standard form and the other in factored form. Find the missing numbers.

1. $x^2 + 9x + 14$ and $(x + 2)(x + \Box)$
2. $x^2 - 9x + 20$ and $(x - \Box)(x - \Box)$
3. $2x^2 + 2x - 24$ and $2(x + \Box)(x - 3)$
4. $\Box x^3 + 11x^2 - 17x + 6$ and $(-x + 3)(2x - 1)(x - 2)$
5. $6x^3 + 2x^2 - 16x + 8$ and $(x - 1)(2x + 4)(\Box x - 2)$
6. $2x^3 + 7x^2 - 7x - 12$ and $(2x - 3)(x + \Box)(x + \Box)$
7. $x^3 + 6x^2 + \Box x - 10$ and $(x + 2)(x - 1)(x + \Box)$

Lesson 13 Summary

In earlier grades, we learned how to add, subtract, and multiply numbers. We also learned that one way to divide numbers, like 1573 divided by 11, is by using long division.

$$\begin{array}{r} 1 \\ 11\,\overline{)\,1573} \\ \underline{11} \\ 4 \end{array} \qquad \begin{array}{r} 14 \\ 11\,\overline{)\,1573} \\ \underline{11} \\ 47 \\ \underline{44} \\ 3 \end{array} \qquad \begin{array}{r} 143 \\ 11\,\overline{)\,1573} \\ \underline{11} \\ 47 \\ \underline{44} \\ 33 \\ \underline{33} \\ 0 \end{array}$$

Here the division has been completed in stages, focusing on the highest power of 10 (1,000) in the dividend 1,573, and working down. This long division shows that $1573 = (11)(143)$.

Similar to integers, we can add, subtract, and multiply polynomials. It turns out that we can also use long division on polynomials. Instead of focusing on powers of 10, in polynomial long division we focus on powers of x. Just as we started with the highest power or 10, we start with the highest power of x, the leading term, and work down to the constant term. For example, here is $x^3 + 5x^2 + 7x + 3$ divided by $x + 1$ completed in three stages. Notice how terms of the same degree are in the same columns.

$$\begin{array}{r} x^2 \\ x+1\,\overline{)\,x^3 + 5x^2 + 7x + 3} \\ \underline{-x^3 - x^2} \\ 4x^2 + 7x \end{array} \qquad \begin{array}{r} x^2 + 4x \\ x+1\,\overline{)\,x^3 + 5x^2 + 7x + 3} \\ \underline{-x^3 - x^2} \\ 4x^2 + 7x \\ \underline{-4x^2 - 4x} \end{array} \qquad \begin{array}{r} x^2 + 4x + 3 \\ x+1\,\overline{)\,x^3 + 5x^2 + 7x + 3} \\ \underline{-x^3 - x^2} \\ 4x^2 + 7x \\ \underline{-4x^2 - 4x} \\ 3x + 3 \end{array}$$

At each stage, the focus is only on the term with the largest exponent that's left. At the conclusion, we can see that $x^3 + 5x^2 + 7x + 3 = (x + 1)(x^2 + 4x + 3)$.

Lesson 13 Practice Problems

1. The polynomial function $B(x) = x^3 - 21x + 20$ has a known factor of $(x - 4)$. Rewrite $B(x)$ as a product of linear factors.

2. Let the function P be defined by $P(x) = x^3 + 7x^2 - 26x - 72$ where $(x + 9)$ is a factor. To rewrite the function as the product of two factors, long division was used but an error was made:

$$
\begin{array}{r}
x^2 + 16x + 118 \\
x + 9 \overline{\smash{\big)}\, x^3 + 7x^2 - 26x - 72} \\
\underline{-x^3 + 9x^2 } \\
16x^2 - 26x \\
\underline{-16x^2 + 144x } \\
118x - 72 \\
\underline{-118x + 1062} \\
990
\end{array}
$$

How can we tell by looking at the remainder that an error was made somewhere?

3. For the polynomial function $A(x) = x^4 - 2x^3 - 21x^2 + 22x + 40$ we know $(x - 5)$ is a factor. Select **all** the other linear factors of $A(x)$.

A. $(x + 1)$

B. $(x - 1)$

C. $(x + 2)$

D. $(x - 2)$

E. $(x + 4)$

F. $(x - 4)$

G. $(x + 8)$

4. Match the polynomial function with its constant term.

A. $P(x) = (x - 2)(x - 3)(x + 7)$

B. $P(x) = (x + 2)(x - 3)(x + 7)$

C. $P(x) = \frac{1}{2}(x - 2)(x - 3)(x + 7)$

D. $P(x) = 5(x - 2)(x - 3)(x + 7)$

E. $P(x) = -5(x - 2)(x - 3)(x + 7)$

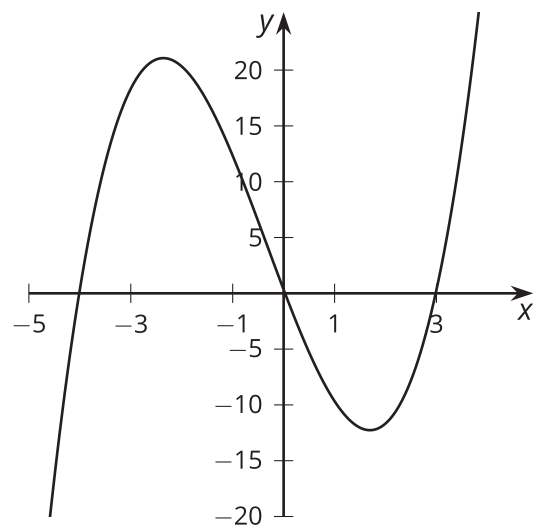
1. -210
2. -42
3. 21
4. 42
5. 210

(From Unit 2, Lesson 6.)

5. What are the solutions to the equation $(x - 2)(x - 4) = 8$?

(From Unit 2, Lesson 11.)

6. The graph of a polynomial function f is shown. Which statement is true about the end behavior of the polynomial function?



A. As x gets larger and larger in the either the positive or the negative direction, $f(x)$ gets larger and larger in the positive direction.

B. As x gets larger and larger in the positive direction, $f(x)$ gets larger and larger in the positive direction. As x gets larger and larger in the negative direction, $f(x)$ gets larger and larger in the negative direction.

C. As x gets larger and larger in the positive direction, $f(x)$ gets larger and larger in the negative direction. As x gets larger and larger in the negative direction, $f(x)$ gets larger and larger in the positive direction.

D. As x gets larger and larger in the either the positive or negative direction, $f(x)$ gets larger and larger in the negative direction.

(From Unit 2, Lesson 8.)

7. The polynomial function $p(x) = x^3 + 3x^2 - 6x - 8$ has a known factor of $(x + 4)$.

a. Rewrite $p(x)$ as the product of linear factors.

b. Draw a rough sketch of the graph of the function.

(From Unit 2, Lesson 12.)