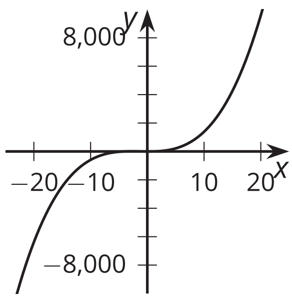
Lesson 8: End Behavior (Part 1)

- Let's investigate the shape of polynomials.

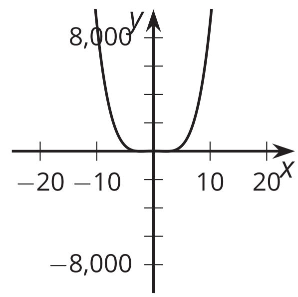
8.1: Notice and Wonder: A Different View

What do you notice? What do you wonder?

$y = x^3 + 4x^2 - x - 4$



$y = x^4 - 10x^2 + 9$



8.2: Polynomial End Behavior

1. For your assigned polynomial, complete the column for the different values of x. Discuss with your group what you notice.

x	$y = x^2 + 1$	$y = x^3 + 1$	$y = x^4 + 1$	$y = x^5 + 1$
-1000				
-100				
-10				
-1				
1				
10				
100				
1000				

2. Sketch what you think the **end behavior** of your polynomial looks like, then check your work using graphing technology.

Are you ready for more?

Mai is studying the function $p(x) = -\frac{1}{100}x^3 + 25{,}422x^2 + 8x + 26$. She makes a table of values for p with $x = \pm 1, \pm 5, \pm 10, \pm 20$ and thinks that this function has large positive output values in both directions on the x-axis. Do you agree with Mai? Explain your reasoning.

8.3: Two Polynomial Equations

Consider the polynomial $y = 2x^5 - 5x^4 - 30x^3 + 5x^2 + 88x + 60$.

1. Identify the degree of the polynomial.

2. Which of the 6 terms, $2x^5$, $5x^4$, $30x^3$, $5x^2$, $88x$, or 60, is greatest when:
    a. $x = 0$

    b. $x = 1$

    c. $x = 3$

    d. $x = 5$

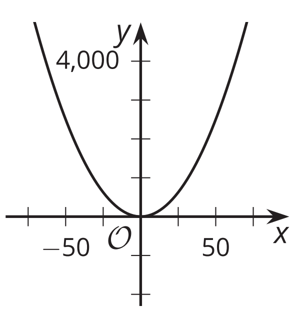
3. Describe the end behavior of the polynomial.

Lesson 8 Summary

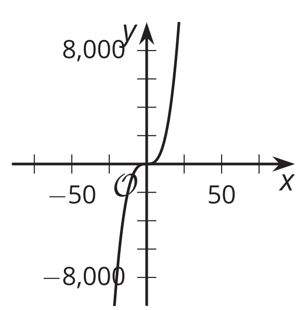
We know that if the expression for a polynomial function f written in factored form has the factor $(x - a)$, then a is a zero of f (that is, $f(a) = 0$) and the point $(a, 0)$ is on the graph of the function. But what about other values of x? In particular, as we consider values of x that get larger and larger in either the negative or positive direction, what happens to the values of $f(x)$?

The answer to this question depends on the degree of the polynomial, because any negative real number raised to an even power results in a positive number. For example, if we graph $y = x^2$, $y = x^3$ and $y = x^4$ and zoom out, we see the following:

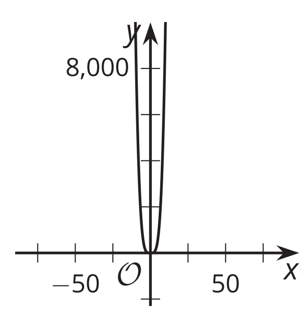
$y = x^2$



$y = x^3$



$y = x^4$



For both $y = x^2$ and $y = x^4$, large positive values of x or large negative values of x each result in large positive values of y. But for $y = x^3$, large positive values of x result in large positive values of y, while large negative values of x result in large negative values of y.

Consider the polynomial $P(x) = x^4 - 30x^3 - 20x^2 + 1000$. The leading term, x^4, almost seems smaller than the other 3 terms. For certain values of x, this is even true. But, for values of x far away from zero, the leading term will always have the greatest value. Can you see why?

x	x^4	$-30x^3$	$-20x^2$	1000	$P(x)$
-500	62,500,000,000	3,750,000,000	-5,000,000	1,000	66,245,001,000
-100	100,000,000	30,000,000	-200,000	1,000	129,801,000
-10	10,000	30,000	-2,000	1,000	39,000
0	0	0	0	1,000	1000
10	10,000	-30,000	-2,000	1,000	-21,000
100	100,000,000	-30,000,000	-200,000	1,000	69,801,000
500	62,500,000,000	-3,750,000,000	-5,000,000	1,000	58,745,001,000

The value of the leading term x^4 determines the **end behavior** of the function, that is, how the outputs of the function change as we look at input values farther and farther from 0. In the case of $P(x)$, as x gets larger and larger in the positive and negative directions, the output of the function gets larger and larger in the positive direction.

Glossary

- end behavior

Lesson 8 Practice Problems

1. Match each polynomial with its end behavior. Some end behavior options may not have a matching polynomial.

    A. $f(x) = 2x^3 + 3x^4 + x^2 - 1$

    B. $f(x) = 1 - 3x + x^2$

    C. $f(x) = 9 + x^4$

    D. $f(x) = 2x + 5$

    1. As x gets larger and larger in either the positive or negative direction, $f(x)$ gets larger and larger in the positive direction.
    2. As x gets larger and larger in the positive direction, $f(x)$ gets larger and larger in the positive direction. As x gets larger and larger in the negative direction, $f(x)$ gets larger and larger in the negative direction.
    3. As x gets larger and larger in the positive direction, $f(x)$ gets larger and larger in the negative direction. As x gets larger and larger in the negative direction, $f(x)$ gets larger and larger in the positive direction.
    4. As x gets larger and larger in either the positive or negative direction, $f(x)$ gets larger and larger in the negative direction.

2. Which polynomial function gets larger and larger in the negative direction as x gets larger and larger in the negative direction?

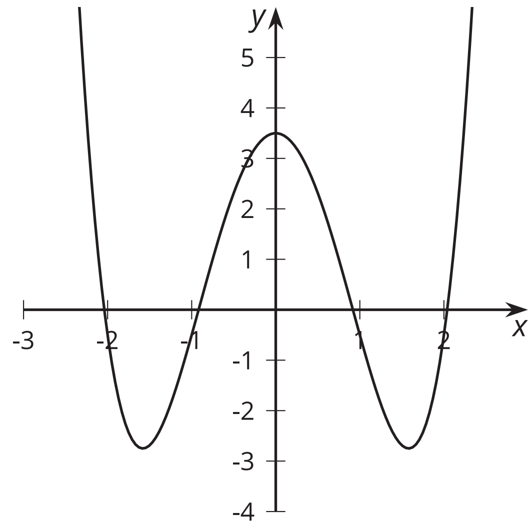
    A. $f(x) = 5x^2 - 2x + 1$

    B. $f(x) = 6x^3 + 4x^2 - 15x + 32$

    C. $f(x) = 7x^4 - 2x^3 + 3x^2 + 8x - 10$

    D. $f(x) = 8x^6 + 1$

3. The graph of a polynomial function f is shown. Which statement about the polynomial is true?



A. The degree of the polynomial is even.

B. The degree of the polynomial is odd.

C. The constant term of the polynomial is even.

D. The constant term of the polynomial is odd.

4. Andre wants to make an open-top box by cutting out corners of a 22 inch by 28 inch piece of poster board and then folding up the sides. The volume $V(x)$ in cubic inches of the open-top box is a function of the side length x in inches of the square cutouts.

   a. Write an expression for $V(x)$.

   b. What is the volume of the box when $x = 6$?

   c. What is a reasonable domain for V in this context?

(From Unit 2, Lesson 1.)

5. For each polynomial function, rewrite the polynomial in standard form. Then state its degree and constant term.

   a. $f(x) = (3x + 1)(x + 2)(x - 3)$

   b. $g(x) = \text{-}2(3x + 1)(x + 2)(x - 3)$

(From Unit 2, Lesson 6.)

6. Kiran wrote $f(x) = (x - 3)(x - 7)$ as an example of a function whose graph has x-intercepts at $x = \text{-}3, \text{-}7$. What was his mistake?

(From Unit 2, Lesson 7.)

7. A polynomial function, $f(x)$, has x-intercepts at $(\text{-}6, 0)$ and $(2, 0)$. What is one possible factor of $f(x)$?

(From Unit 2, Lesson 7.)

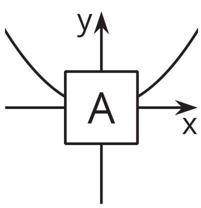
Lesson 9: End Behavior (Part 2)

- Let’s describe the end behavior of polynomials.

9.1: It’s a Cover Up

Match each of the graphs to the polynomial equation it represents. For the graph without a matching equation, write down what must be true about the polynomial equation.

A



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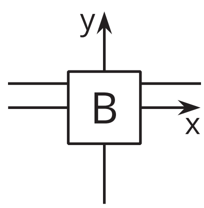

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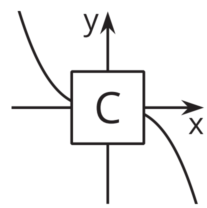

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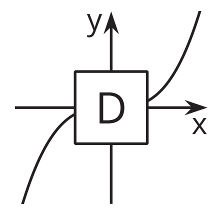


1. $y = x(x+3)(2-x)$
2. $y = 1 - 3x + 5x^4$
3. $y = 5(x+3) - 5x$

9.2: The Case of Unexpected End Behavior

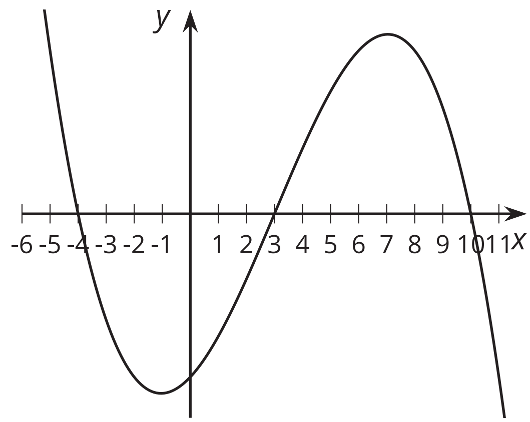
1. Write an equation for a polynomial with the following properties: it has even degree, it has at least 2 terms, and, as the inputs get larger and larger in either the negative or positive directions, the outputs get larger and larger in the negative direction.

   Pause here so your teacher can review your work.

2. Write an equation for a polynomial with the following properties: it has odd degree, it has at least 2 terms, as the inputs get larger and larger in the negative direction the outputs get larger and larger in the positive direction, and as the inputs get larger and larger in the positive direction, the outputs get larger and larger in the negative direction.

Are you ready for more?

In the given graph all of the horizontal intercepts are shown. Find a function with this general shape and the same horizontal intercepts.



9.3: Which is Greater?

M and N are each functions of x defined by $M(x) = -x^3 - 2x + 8$ and $N(x) = -20x^2 + 3x + 8$.

1. Describe the end behavior of M and N.

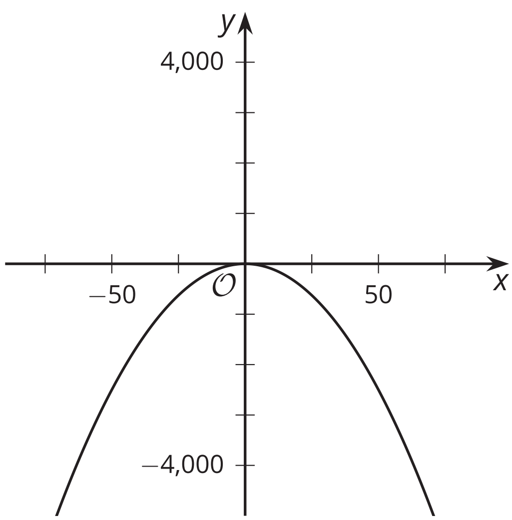
2. For $x > 0$, which function do you think has greater values? Be prepared to share your reasoning with the class.

Lesson 9 Summary

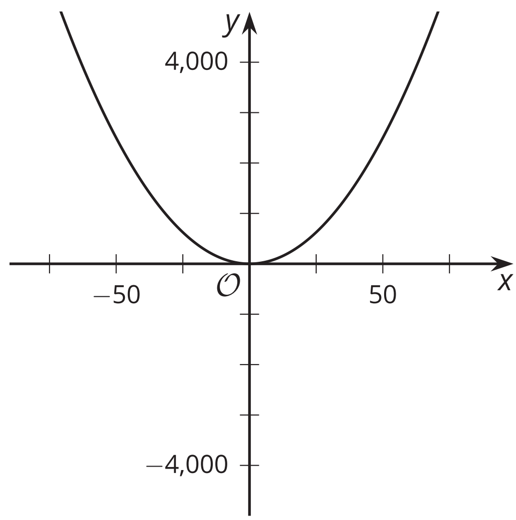
What happens when we multiply a number by a negative number? If the original number was positive, the product is negative. But if the original number was negative, the product is positive. The sign of the new number is the opposite of the original number.

Now let's consider the polynomial functions $f(x) = x^2$ and $g(x) = -x^2$. For any non-zero real number x, the output of f is positive while the output of g is negative. The signs of all the output values for g are the opposite of those of f. The difference between these two functions is also easy to see when we look at their graphs.

$g(x) = -x^2$



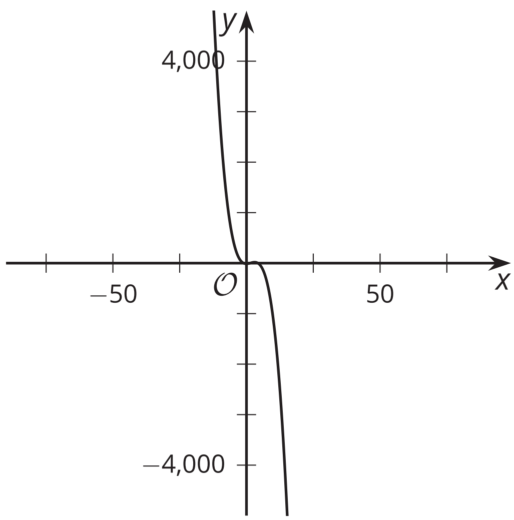
$f(x) = x^2$



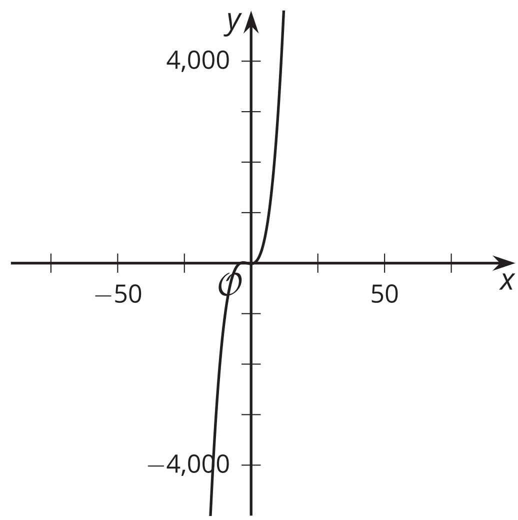
This is the effect of a negative leading coefficient: the end behavior of the polynomial is the opposite of what it would be if the leading coefficient were positive. For polynomials of odd degree, we can see that a negative leading coefficient has the same effect on the end behavior.

Here are the graphs of $y = \text{-}(x-1)(2x+3)(x+4)$, which has a leading term of $\text{-}2x^3$, and $y = (x-1)(2x+3)(x+4)$, which has a leading term of $2x^3$. They have the same zeros, but opposite end behavior, because they have opposite signs on their leading coefficients.

$y = \text{-}(x-1)(2x+3)(x+4)$



$y = (x-1)(2x+3)(x+4)$



Lesson 9 Practice Problems

1. Match the polynomial with its end behavior.

A. $f(x) = -2x + 3$

B. $f(x) = x^2 - 6x + 3$

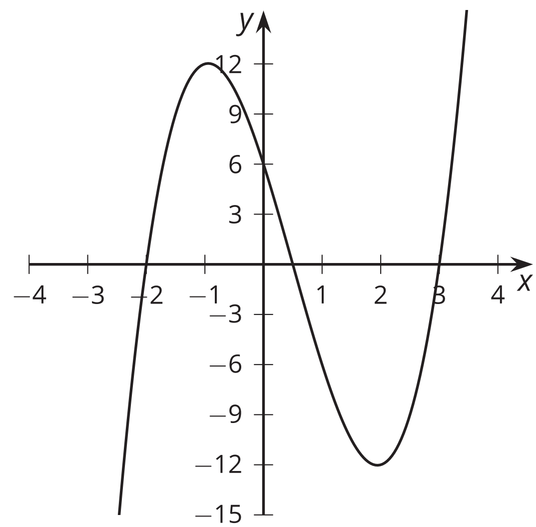
C. $f(x) = 1 - x^2 + 2x^3$

D. $f(x) = 7 - x^4$

1. As x gets larger and larger in either the positive or negative direction, $f(x)$ gets larger and larger in the positive direction.
2. As x gets larger and larger in the positive direction, $f(x)$ gets larger and larger in the positive direction. As x gets larger and larger in the negative direction, $f(x)$ gets larger and larger in the negative direction.
3. As x gets larger and larger in the positive direction, $f(x)$ gets larger and larger in the negative direction. As x gets larger and larger in the negative direction, $f(x)$ gets larger and larger in the positive direction.
4. As x gets larger and larger in either the positive or negative direction, $f(x)$ gets larger and larger in the negative direction.

2. State the degree and end behavior of $f(x) = -x^3 + 5x^2 + 6x + 1$. Explain or show your reasoning.

3. The graph of a polynomial function f is shown. Select **all** the true statements about the polynomial.



A. The degree of the polynomial is even.

B. The degree of the polynomial is odd.

C. The leading coefficient is positive.

D. The leading coefficient is negative.

E. The constant term of the polynomial is positive.

F. The constant term of the polynomial is negative.

4. Write the sum of $5x^2 + 2x - 10$ and $2x^2 + 6$ as a polynomial in standard form.

(From Unit 2, Lesson 4.)

5. State the degree and end behavior of $f(x) = 4x^3 + 3x^5 - x^2 - 2$. Explain or show your reasoning.

(From Unit 2, Lesson 8.)

6. Select **all** the polynomial functions whose graphs have x-intercepts at $x = 4, \text{-}\frac{1}{4}, \text{-}2$.

A. $(x+4)(4x-1)(x-2)$

B. $(x-4)(4x+1)(x+2)$

C. $(x-4)(4x-1)(x-2)$

D. $(x+4)(4x+1)(x+2)$

E. $(2x+4)(4x-1)(x-2)$

F. $(4x-16)(4x+1)(x+2)$

(From Unit 2, Lesson 7.)